COMPETING FOR PRIORITIES IN SCHOOL CHOICE

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Abstract. We present a model in which students can influence their priorities in a school choice mechanism through a first-stage contest. We show that efficiency improvements at the level of the mechanism can lead to net efficiency losses if they come at the price of increased allocative inequalities, increasing competition in the contest. We apply these results to the deferred and immediate acceptance mechanisms (DA and IA) and show that, even when DA is more allocatively efficient, IA may remain more efficient overall because it features less inequality between students with high and low priorities.

Keywords: School Choice; Contests; Matching

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1. Introduction

Students in New York City who want to attend one of the City's most prestigious high schools such as Stuyvesant or Bronx Science must take the Specialized High School Admissions Test. The results of this test influence the placements by determining students' priorities in the subsequent mechanism that assigns students to schools. Students who score higher on the placement exam receive higher priority in the procedure, and a higher priority tends to yield a seat in a more preferred school. Since these schools use a sequential dictator mechanism, the top-scoring student receives her top choice. The 100th best scorer only receives a seat in her most favored school among those that the 99th best scorers have not yet filled (Dobbie and Roland G. Fryer, 2011).

In other places, like Boston, students gain priority based on several criteria, such as whether the student lives within walking distance to the school (Abdulkadiroğlu et al., 2005b). This implies that students compete for access to better schools, in part, by relocating near their favorite schools, as evidenced by the effect of school quality on house prices (Crone, 1998; Downes and Zabel, 2002; Leech and Campos, 2003; Reback, 2005; Gibbons and Machin, 2008). Again, higher priorities in better schools are useful only as far as the assignment mechanism uses these priorities. For example, Boston changed its school choice mechanism in 2005 to *deferred acceptance* (DA) which gives substantial importance to priority. This contrasts with the previous use of the *immediate acceptance* (IA) mechanism where priority plays a lesser role.

The matching mechanism itself is an essential component in the overall assignment process. The matching literature has made significant progress in analyzing the properties of mechanisms. Abdulkadiroglu and Sönmez (2003) and Pathak (2011) provide a review of results in school choice.[1] Famously, these results led to the redesign mentioned above for the assignment procedure in Boston and the redesign of the broader New York Public School assignment procedures in 2003 (Abdulkadiroğlu et al., 2005). Researchers generally analyze mechanisms by assuming that the priorities are fixed and exogenous. In this paper, we emphasize that the interaction between the matching mechanism and the "game" through which priorities are determined can be an important consideration, in particular for the overall efficiency of the assignment procedure.

To illustrate the importance of this interaction, suppose that a school district allocates some proportion of its seats through an algorithm based on priority and allocates the remaining seats "at random" (in a way that is independent of priorities).[2] Further, suppose that priorities at a school are a continuous function of the distance a student lives from the school. This is the case, for example, in the French-speaking region of

[1] See Sönmez and Ünver (2011) for a broader overview.

[2] In practice, it is common for school districts to allocate different layers of seats — or seats at different schools — through different mechanisms. Examples include New York City, which uses a different mechanism for elite high schools than for the remaining high schools in the district (Abdulkadiroğlu et al., 2005a; Dobbie and Roland G. Fryer, 2011).

Belgium, where one of the factors affecting priority is the distance between one's residence and the school at stake (Cantillon, 2015). As the proportion of seats allocated based on priority increases, incentives to relocate closer to one's favorite schools increase.[3] However, relocation may imply choosing a house and a neighborhood that would not be optimal without this priority consideration. Thus, to the extent that the resulting relocation is wasteful, welfare could be negatively affected by an increase in the proportion of seats allocated based on priority.

Of course, the overall efficiency picture is more complicated. In particular, the effect of a change in mechanism on the wasteful competition for priorities must be weighed against the possible gains in allocative efficiency brought about by that change.[4] To study overall efficiency, we embed a simplified matching environment into a two-stage game with a first-round contest for priority (the "*effort stage*"). This allows us to formally analyze the effect of changes to the matching mechanism on overall efficiency by including interactions between the mechanism and preliminary effort.

This approach yields important insights. We show that efficiency improvements at the level of the mechanism can lead to net efficiency *losses* through increased competition in the effort stage (by "net" efficiency, we mean efficiency at the level of the procedure as a whole, including the effort stage). All else equal, net efficiency increases with the efficiency of the mechanism, but decreases with its allocative inequality because increases in the latter induce fiercer competition and more wasteful effort, at the level of the effort stage. The reduction of allocative inequality at the level of the mechanism can therefore be justified by net efficiency considerations. Improvements in allocative efficiency can also have a detrimental effect on net efficiency if they come at the price of increased allocative inequalities. In the context of our simplified matching environment, this implies that even when DA is more allocatively efficient than IA, the latter may remain more efficient overall because it features less inequality between students with high- and low-priority (in equilibrium).

Although our results apply specifically to school choice and the IA and DA mechanisms, they point, more broadly, to the importance of considering matching mechanisms in a general equilibrium setting when costly effort is involved in influencing potential outcomes.

[3] Reback (2005) show empirically that a change in the public school assignment procedure in Minnesota in the early 1990s (namely the adoption of a public school choice program) led to significant changes in home prices. Priority access to schools caused these price movements.

[4] In the above example, if increasing x also makes the allocation of seats more efficient, this increase in efficiency may overwhelm the decrease in efficiency caused by fiercer competition for priorities.

1.1. **Related literature.**

Our paper is the first, to our knowledge, to consider the effect of a matching mechanism on the incentives for players to compete for priority. However, several papers are related in terms of our general approach.[5]

Hatfield et al. (2018) considers a general matching / allocation environment in which players make investment decisions that impact their valuations at the matching stage (with potential uncertainty about the impact of investment). As in our paper, the authors consider the effect of the matching mechanism on overall efficiency, including prematching investment. They show that a mechanism incentivizes efficient investment if and only if it is allocatively efficient and strategy-proof, generalizing previous results of Rogerson (1992) and Bergemann and Välimäki (2002). In Section 4.2, we also consider the case where a player's investments affect their preferences. However, in contrast to Hatfield et al. (2018) we do not allow transfers, and the investments of our players affect how the allocation mechanism treats them.

The role of investments in Hsu (2016) mirrors our approach. There, students are endowed with an allotment of time to devote to studying various topics. These investments determine a student's priority in the matching stage, but not their valuation. In contrast to our paper, Hsu (2016) analyzes the effect of the choice of mechanism on the incentives for students to spend time on topics for which they have a particular talent or pursue a diversity of abilities. We focus on a more traditional welfare analysis.

Hatfield et al. (2016) is closely related in theme to our work, but takes a different approach to analyze competition. In their model, investments by schools affect students' preferences. However, instead of studying the welfare effect of investments as we do, they introduce an ordinal criterion that implies that schools have an incentive to invest in improvements. They show that, in large school districts, this condition is approximately met when the matching mechanism is stable.

Avery and Pathak (2021) study the interaction between school choice and the housing market. Similarly to our results, they demonstrate that the improvements of a mechanism may be undermined through distortions of incentives at an earlier stage. Although school choice is intended to improve the availability of high-quality schools, in their model, introducing school choice causes changes in housing prices that lead both high- and low-income families to move out of districts with school choice. This "flight" can produce worse outcomes for low-income families.

A final related paper is Zhang (2020), which demonstrates that stable two-sided matching mechanisms can induce players to engage in risky gambles for desirable characteristics in a prematching stage.

1.2. **Paper structure.**

Our paper is structured as follows. In Section 2, we describe our model and provide details of the matching environment and the prematching effort

[5]Students' behavior in our model, which leads to decreases in net welfare despite improvements to the allocation of schools, which is a specific case of rent-seeking. See, for instance, Tullock (1967); Krueger (1974); Buchanan et al. (1980); Tullock et al. (1993); Congleton et al. (2008); Tollison (2012).

stage. In Section 3, we demonstrate that the effort stage constitutes a *generic* all-pay contest and leverage the results of Siegel (2009) to derive comparative statics which relate properties of the matching mechanism to overall welfare in the equilibrium of the two-stage effort/matching game. In Section 4 we use these results to study the overall efficiency of popular school choice mechanisms such as *deferred acceptance* and *immediate acceptance* under various conditions. We conclude with a discussion of our results and potential extensions in section 5.

2. Model

Our model is split into two main stages. In the effort stage, students compete for priorities in a contest. In the second stage, these priorities are used in a school choice mechanism to determine seats at schools. The timing of our model is as follows. Students first compete in a contest by simultaneously choosing effort under common knowledge of effort costs. Students then learn their rank in the contest and how these ranks translate into priorities at schools. Students also learn their cardinal preferences over schools (but not the cardinal utility of other students) before reporting ordinal preferences over schools in the second-stage school choice mechanism. We discuss some aspects of the timing and information structure in more detail in section 2.3. Since the effort stage's incentives depend on the matching mechanism's potential outcomes, it is convenient to describe the matching stage first.

2.1. **Matching stage.** The model of the matching stage is similar to Abdulkadiroğlu et al. (2011). There are $m \geq 2$ schools, $S := \{s_1, \dots, s_m\}$ with the index set $A := \{1, \dots, m\}$. School $s_a \in S$ has capacity $q_a > 0$. Departing from Abdulkadiroğlu et al. (2011), we divide the schools into two groups. The set of "high-quality" schools is $S^h := \{s_1, \dots, s_g\}$ for some $g \in \{1, \dots, m-1\}$ and the set of "low-quality" schools is $S^\ell := \{s_{g+1}, \dots, s_m\}$.

Schools in S can be attended by $n \geq 2$ students. Among these students, h have a high **priority type** which results in a high-priority at schools s_1 to s_g (these are the h students who exert the highest effort in the initial contest stage, see below).[6] Priorities are otherwise determined via a symmetric tie-breaking rule.[7] This simple priority structure can be viewed as an approximation of school districts where only a few strict priorities are awarded (e.g., based on whether a student lives in the "walking zone" of a school) with ties otherwise broken at random, which is common in practice. Throughout, we assume that $\sum_{a\in\{1,\dots,g\}} q_a = h$ and $\sum_{a\in\{g+1,\dots,m\}} q_a = \ell$. That is, the total capacity of

[6]Slightly abusing the notation, we use ℓ and h to denote both the priority types themselves (low or high) and the number of students of a certain priority type (ℓ low-priority types and h high-priority types).

[7]This means the h high-priority types always have a higher priority at the g high-quality schools than the ℓ low-priority types. Priorities at high-quality schools among the h high-priority types and the ℓ low-priority types are determined symmetrically at random (where "symmetrically" we mean that each student has the same probability of occupying any priority rank at a high-quality school than any other student of her priority type). Priorities in low-quality schools are also determined symmetrically at random but among *all* students, low- and high-priority types alike.

high-quality schools is equal to the number of high-priority type students, and the total capacity of the low-quality schools is equal to the number of low-priority type students.

Each student draws vNM utility values $\boldsymbol{v} = (v_1, \ldots, v_m)$ for the schools from the distribution $f_p(\boldsymbol{v})$, where $f_p(\boldsymbol{v})$ may depend on the students' priority type $p \in \{\ell, h\}$.[8] We refer to $\boldsymbol{v}$ as a student's **valuation type**. Students compete for higher priorities through the effort stage described below. *After* acquiring a certain priority type through the effort stage, students learn their valuation type (but not the valuation type of other students) and report ordinal preferences over schools. Assignment to a school is determined based on priorities and reported preferences by some school choice mechanism X. This paper focuses on deferred acceptance (DA) and immediate acceptance mechanisms (IA). We refer to the extensive school choice literature for a detailed description of these mechanisms (e.g., Abdulkadiroglu and Sönmez, 2003).

Since students learn their utility value *after* the effort stage (see Section 2.3), in the effort stage, students have identical ex ante utilities for securing a high- or a low-priority type. We denote these ex ante utilities by V_h and V_ℓ, or sometimes V_h^X and V_ℓ^X when a reference is made to the mechanism X the equilibrium of which induces utilities V_h and V_ℓ. The expected utilities V_ℓ and V_h are the utilities on which students base their decisions in the effort stage. For consistency with the effort stage and to avoid possible extreme cases, we always assume $V_h \geq V_\ell$.

2.2. **Effort stage.** Students compete by putting effort into acquiring the characteristics that result in higher priorities. In the context of school choice, this can mean moving closer to a particular set of schools or spending time and money on preparing for a standardized test.

The **total effort** student i puts into obtaining a higher priority is t_i. Student i's total effort is decomposed into i's **priority-independent effort** a_i and i's **priority-dependent effort** r_i. As its name suggests, we think of a_i as i's optimal effort given all benefits independent of acquiring a higher matching priority. For example, a_i could represent the amount of effort student i would put into preparing for standardized tests "anyways", for reasons other than the effect of test-results on priorities at schools (e.g., social prestige, curiosity, and use in determining outcomes independent of school assignments). Priority-independent effort can also represent the effort a family would typically put into moving to a new neighborhood for all reasons other than increased priority at schools in this neighborhood. In contrast, **priority-dependent effort** r_i is the part of the total effort exerted due to the effect of effort on school priorities. For example, r_i could represent the additional cost a family is willing to pay to relocate to a new neighborhood with desirable schools specifically because of the higher priority that the family would secure in these desirable schools.

[8]We later impose constraints on the support of $f_p(\boldsymbol{v})$ that formalize the idea that schools in $\{s_1, \ldots, s_g\}$ are of higher quality than schools in $\{s_{g+1}, \ldots, s_m\}$.

Types	Notation	Description	Timing
Effort type	a	Baseline effort a student exerts regardless of its effect on priority types	Exogenously given at the beginning of the effort stage
Priority type	p	Whether the student is high or low-priority at high-quality schools	Acquired through the effort stage
valuation type	$\boldsymbol{v}$	Utility value of being assigned to a high-quality school	Drawn from distribution $f_p(\boldsymbol{v})$ after the effort stage

TABLE 1. Summary of the three type-dimensions that characterize students through the game and when students learn about them.

Because priority-independent effort is chosen optimally for reasons other than securing higher priorities, we take a_i as given for every student and model it as a sunk cost. In our effort stage, students choose a level of total effort t_i that is at least a_i. Whereas priority-dependent effort is a decision variable, priority-independent effort is an exogenous characteristic. Therefore, we often refer to a student's priority-independent effort as her **effort type**. A summary of the three type dimensions that characterize students through the game (that is, effort type, priority type, and valuation type) is provided in Table 1. Index players by their effort type: $a_1 \geq a_2 \geq a_3 ... \geq a_n$. We assume that $a_{h+1} \neq a_j$ for $j \neq h+1$.[9]

Higher effort is costly and induces no benefit other than the higher priorities it may provide (recall that any "intrinsic value" of effort is already captured by a_i). The cost of total effort t_i for student i is given by $e(t_i)$. The cost of additional priority-dependent effort is $e(t_i) - e(a_i)$ which can also be written $e(r_i + a_i) - e(a_i)$. We assume that e is continuous, strictly increasing, and there exists some $\tilde{t}$ such that $V_h - V_\ell < e(\tilde{t}) - e(a_1)$, which guarantees that for all students, there is some level of effort that cannot be justified even if it guarantees high-priority.

If a student exerts one of the h highest total efforts, she becomes a high-priority student and her utility is $V_h - (e(t_i) - e(a_i))$. Otherwise, she is a low-priority student and her utility is $V_\ell - (e(t_i) - e(a_i))$. Given a vector of total effort $\boldsymbol{t}$, let $H_i(\boldsymbol{t})$ be probability student i becomes high-priority. Define H_i as follows: $\sum_{i=1}^{n} H_i(\boldsymbol{t}) = h$ and,

$$H_i(\boldsymbol{t}) = \begin{cases} 0 & t_i < t_j \text{ for at least } h \text{ players } j \neq i \\ 1 & t_i > t_j \text{ for at least } n-h \text{ players } j \neq i \\ \text{any value in } [0,1] & \text{otherwise} \end{cases}.$$

[9]This assumption is required for the model to meet the *power condition* of Siegel (2009).

Student i's utility given total effort vector $\boldsymbol{t}$ is:

$$u_i(\boldsymbol{t}) := H_i(\boldsymbol{t})\Big[V_h - \big(e(t_i) - e(a_i)\big)\Big] + \big(1 - H_i(\boldsymbol{t})\big)\Big[V_\ell - \big(e(t_i) - e(a_i)\big)\Big].$$

If t^* is an equilibrium of the effort stage, the **net welfare** when ex ante utilities are V_ℓ and V_h and effort types are $\boldsymbol{a}$ is[10]

$$W(V_h, V_\ell, \boldsymbol{a}) := \sum_{i=1}^{n} u_i(\boldsymbol{t}^*)$$

2.3. **Discussion of model assumptions.** At least two features of our model deserve comments. First, the game associated with the effort stage is one of complete information, whereas we model the matching stage as an incomplete information game. The latter is desirable because it is hard to conceive of realistic (especially large-scale) school choice problems where every student knows every other student's valuation type. However, one could argue that the same is true of effort types.

As the equilibrium analysis reveals, coordination on a (complete information) Nash equilibrium in the effort stage requires much less than full knowledge of the effort type profile. To select her equilibrium priority-dependent effort, a student only needs to know the threshold effort that is required to deter low effort types from competing for higher priorities.[11] For example, if priority types are awarded based on exam scores, knowing the threshold-scores that prevailed in previous years could provide a reasonably accurate estimate of the current year's threshold, and students might be able to coordinate on the Nash equilibrium.

Second, recall that the students learn their valuation type *after* learning the priority type they acquired through the effort stage. The distribution from which valuations are drawn may depend on the acquired priority type, which students know.[12] But all students compete for higher priorities before knowing the realization of these (possibly priority type specific) distributions. In particular, students know they will prefer high-quality schools to low-quality ones, but *not* exactly *how intense* this preference will be.

Although students' *ordinal* ranking of schools may be relatively stable over time, students may, in some cases, learn their precise *cardinal* values for schools after acquiring priorities. For example, this is the case; if to secure a higher priority at a neighborhood's school, students have to move to that neighborhood years in advance of applying to these schools.

[10]Dependence of $W(V_h, V_\ell, \boldsymbol{a})$ on a particular equilibrium $\boldsymbol{t}^*$ is omitted in the notation because we show below (Theorem 1) that equilibrium utilities are unique in the effort stage.

[11]In turn, determining the threshold effort requires knowing only the h-th highest effort type, which is the only information on the effort type profile that is required for students to coordinate on a (complete information) Nash equilibrium in the effort stage.

[12]E.g., if a student obtained a higher priority at a school by moving to the school's neighborhood, she may be drawing a utility value for that school from a different distribution than a student who did not move to the school's neighborhood (e.g., because she now lives closer to that school).

More importantly, assuming that students have the same expectations about their future valuation type in the effort stage enables distinguishing between wasteful and signaling aspects of effort. As a first step and a benchmark, it can be useful to isolate the wasteful aspect of effort and compare net welfare under different school mechanisms assuming that effort carries no welfare-relevant information. This can only be done if a higher effort does not signal a more intense preference for high-quality schools. This requires assuming that: (i) in the matching stage, students draw valuation types from the same distribution, regardless of the priority type they acquired in the effort stage, and (ii) in the effort stage, students only know this distribution and do not have additional idiosyncratic information about their future valuation type. We maintain these two assumptions throughout section 4.1.

In Section 4.2, we open the door to the informative nature of effort in terms of preference intensity by relaxing (i) and allowing the distribution of valuation types to depend on priority types. In this section, we maintain the assumption that students only learn the realization of their valuation type after the effort stage (i.e., we essentially maintain (ii)). However, students now anticipate that if they acquire a high-priority type, they will draw a valuation type from a different distribution than if they had acquired a low-priority type. As we show, this is sufficient for effort to become a welfare relevant signal and alter some — but not all — of the net welfare comparisons obtained in section 4.1. Relaxing (ii) and endowing students in the effort stage with idiosyncratic information about their future valuation type may provide further opportunities for effort to become a signal of preference intensity. We discussed this last point in the conclusion and the appendix.

3. Equilibrium in the effort stage

After a normalization to measure utility relative to the value of being low-priority, this model is a *generic* all-pay contest (Siegel, 2009) (see Appendix Section A.1 for a proof). Throughout this section, we use the results of Siegel (2009) to characterize the equilibrium of the effort stage of our model.

3.1. Utility in equilibrium.

3.1. Utility in equilibrium. Let $\Delta = V_h - V_\ell$. We refer to this as *allocative inequality.* Let the *threshold score* of the contest be the $\tilde{t}$ that solves $\Delta = e\left(\tilde{t}\right) - e\left(a_{h+1}\right)$. Thus,

$$\tilde{t} = e^{-1}\left(\Delta + e\left(a_{h+1}\right)\right).$$

Since this contest is *generic*, the expected utilities of the players in any equilibrium of the first-stage effort game are characterized by Siegel (2009, Theorem 1).

Theorem 1. *In any equilibrium with effort type vector $\boldsymbol{a}$, the expected utilities of a player with effort type a_i is:*

$$
(1) \qquad u_i(\boldsymbol{a}) \coloneqq \begin{cases} V_h, & \text{if } i < h+1 \text{ and } a_i > e^{-1}(\Delta + e(a_{h+1})) \\ V_\ell + e(a_i) - e(a_{h+1}), & \text{if } i < h+1 \text{ and } a_i < e^{-1}(\Delta + e(a_{h+1})) \\ V_\ell, & \text{if } i \geq h+1 \end{cases}.
$$

In any equilibrium, the $n-h$ players with the lowest baseline scores do not exert priority-dependent effort and have an expected utility of V_ℓ.[13] This is precisely what they would expect to get if the baseline scores themselves were used to determine priority.

The players with the h highest baseline scores, on the other hand, have expected utility equal to V_h less the minimum effort cost to obtain at least the threshold score. Whether these players exert priority-dependent effort depends on whether or not their baseline score is already above the threshold. We say that the players for which $i < h+1$ are in the **competitive set**, which we denote by $C(\Delta, \boldsymbol{a})$. Formally $C(\Delta, \boldsymbol{a})$ is the set of students $i \in 1, ..., h$ such that $\Delta + e(a_{h+1}) \geq e(a_i)$, which corresponds to the second line of (1). Let $\#C(\Delta, \boldsymbol{a})$ be the size of this set. For given e and Δ, we say that $\boldsymbol{a}$ is **more competitive** the larger $\#C(\Delta, \boldsymbol{a})$.

With the utility characterization in Theorem 1, the net welfare in any equilibrium also has a simple characterization. We first define a few terms. As defined earlier, Δ is the *allocative inequality*, the difference between the expected utility of high- and low-priority players in the second stage. V is the *allocative welfare*. It corresponds to the welfare in a model where additional effort is not possible and baseline scores are used to determine priority. Finally, we define *effort deadweight loss* as $D(\Delta, \boldsymbol{a})$. This is the aggregate welfare cost of effort expended in the effort stage:

Allocative Inequality: $\Delta := V_h - V_\ell$

Allocative Welfare: $V := hV_h + (n-h)V_\ell = nV_\ell + h\Delta$

Effort Deadweight Loss: $D(\Delta, \boldsymbol{a}) = \sum_{i=1}^{h} max\left\{\Delta + e(a_{h+1}) - e(a_i), 0\right\}$

We then have the following characterization.

Corollary 1. *In any equilibrium of the effort stage, expected net welfare is allocative welfare less the effort deadweight loss, i.e.,*

$$
W(V_h, V_\ell, \boldsymbol{a}) = V - D(\Delta, \boldsymbol{a}).
$$

3.2. **Comparative statics.** Allocative welfares V_ℓ and V_h affect net welfare both directly through allocative welfare and indirectly through effort deadweight loss due to changes in competitive incentives. Increases to V_ℓ strictly increase allocative welfare. Increases to V_ℓ also reduce the relative "prize" earned in becoming high-priority. This reduces

[13]This is in terms of unnormalized utility.

competitive incentives, weakly decreasing effort deadweight loss. Thus, changes to V_ℓ strictly increase net welfare.

While increasing V_h also improves allocative welfare, this increase also changes incentives to compete, weakly increasing effort deadweight loss. However, this effect never overwhelms improvements in allocative welfare, and increases in V_h weakly increase net welfare. The improvement is strict as long as some player does not need to put in any extra effort beyond her priority-independent effort to reach the threshold score. That is, if there is a student for which $\Delta + e\left(a_{h+1}\right) < e\left(a_i\right)$. For such a student, priority-independent effort is already enough to guarantee a high priority in equilibrium, since there are not h other students willing to put in the level of effort to attain high-priority with certainty.

Overall, the magnitude of the effect of V_h and V_ℓ on welfare is mediated by the size of the competitive set. The larger the competitive set, the more net welfare increases as V_ℓ increases, and the less it increases as V_h increases. When the competitive set is the entire set of top h students (indexed by priority-independent effort), V_h does not affect net welfare.

All Propositions and Corollaries in this section follow straightforwardly from Theorem 1, and proofs are therefore omitted.

Proposition 1. *For almost all values of* $V_h, V_\ell, \boldsymbol{a}$,

$$\frac{\partial W\left(V_h, V_\ell, \boldsymbol{a}\right)}{\partial V_h} = h - \#C(\Delta, \boldsymbol{a}), \quad \frac{\partial W\left(V_h, V_\ell, \boldsymbol{a}\right)}{\partial V_\ell} = (n-h) + \#C(\Delta, \boldsymbol{a}).$$

Since players are indexed by a_i and e is strictly increasing, all $i < h+1$ are in the competitive set as long as player 1 is in the competitive set. This leads to the following corollary.

Corollary 2. *For any* $V_h, V_\ell, \boldsymbol{a}$*, the net welfare* $W(V_h, V_\ell, \boldsymbol{a})$ *is strictly increasing in* V_ℓ *and increasing in* V_h*. It is strictly increasing in* V_h *if and only if* $\Delta + e\left(a_{h+1}\right) < e\left(a_1\right)$*.*

On the other hand, if even the student with the highest priority independent effort is in the competitive set, the improvements to V_h do not affect welfare since the direct improvements to allocative welfare are completely offset by the increased competitive incentives.

Corollary 3. *For any* $V_h, V_\ell, \boldsymbol{a}$*, if* $\Delta + e\left(a_{h+1}\right) \geq e\left(a_1\right)$*, then* $W(V_h, V_\ell, \boldsymbol{a}) = nV_\ell$ *and* $\frac{\partial W(V_h, V_\ell, \boldsymbol{a})}{\partial V_h} = 0$

In our context, Corollary 3 provides an efficiency justification for fairness considerations. Changes in the allocation mechanism that do not weakly improve the outcome of low types cannot provide robust efficiency gains in highly competitive environments. In other words, for any change to the allocation mechanism that reduces V_ℓ, there is an environment that is sufficiently competitive for that change to reduce net welfare. This is true even if the change increases allocative welfare. For example, it may be tempting to increase V_h by a large amount at the expense of a small decrease in V_ℓ. However, in very competitive environments, the high ability types would compete away almost all of this efficiency improvement such that even a small decrease in value to the low-priority players is enough to cause a net loss in welfare.

This logic is formalized in the following proposition. Again, it shows that an improvement in welfare of the low-priority type is essential to any change in the allocation mechanism aimed at *robustly* improving net welfare (where robustness is with respect to the competitiveness of the effort stage).

Proposition 2. *For any $V_h, V_\ell, \boldsymbol{a}$ and V_h', V_ℓ' such that $V_h > V_h'$ and $V_\ell < V_\ell'$, there exists an $\boldsymbol{a}'$ such that $W(V_h, V_\ell, \boldsymbol{a}) < W(V_h', V_\ell', \boldsymbol{a}')$.*

The previous results emphasize the importance (from an efficiency perspective) of the fate of low-types as competitiveness increases. Even for a fixed, arbitrary level of competitiveness, inequalities between low- and high-types (as represented by Δ) have a significant impact on efficiency. Similarly, fixing allocative welfare, an increase in inequality harms welfare as long as the competitive set is not empty.

Proposition 3. *For any pair of V_h, V_ℓ and V_h', V_ℓ' and any a such that $V(V_h, V_\ell) = V(V_h', V_\ell')$, $\Delta' > \Delta$, and where $\Delta' + e(a_{h+1}) > e(a_h)$, $W(V_h, V_\ell, \boldsymbol{a}) > W(V_h', V_\ell', \boldsymbol{a})$.*

Naturally, although Proposition 3 isolates the inequality effect by imposing $V(V_h, V_\ell) = V(V_h', V_\ell')$, the proposition also has implications for situations where efficiency is improved. Specifically, any efficiency improvement can be offset by a sufficient increase in inequalities.

Corollary 4. *For any pair of V_h, V_ℓ and any $\epsilon > 0$, there exists V_h', V_ℓ' such that $V(V_h', V_\ell') = V(V_h, V_\ell) + \epsilon$ but $\Delta' > \Delta$ and $W(V_h', V_\ell', \boldsymbol{a}) < W(V_h, V_\ell, \boldsymbol{a})$.*

4. Equilibrium in the matching stage

When reporting her preference, a student knows her priority type p as well as her own valuation type $\boldsymbol{v}$, but not the valuation type of other students. The equilibrium probability for a priority type p to be assigned to school s_a when the mechanism is X is denoted by $P_a^{X,p}$. In equilibrium, the expected utility of such a student is then

$$V_{p,\boldsymbol{v}}^X := \sum_{a \in A} v_a P_a^{X,p}.$$

Recall that students do not know the realization of their valuation type during the effort stage. In the effort stage, students only know the distribution $f_p(\boldsymbol{v})$ from which they will draw their valuation type (which, as the subscript indicates, may depend on the priority type p they acquire through the effort stage). Throughout we assume that the supports of f_ℓ and f_h are finite, and we let $\mathcal{V}$ denote a generic set of valuation types that includes the supports of both f_ℓ and f_h.[14] Before knowing the realization of $f_p(\boldsymbol{v})$, the expected utility of a priority type p in the mechanism X is then

$$V_p^X := \sum_{\boldsymbol{v}\in\mathcal{V}} V_{p,\boldsymbol{v}}^X f_p(\boldsymbol{v}).$$

In the effort stage, students base their decision on expected utilities V_ℓ^X and V_h^X. Recall that the allocative welfare of a given mechanism X is then defined as $V^X = hV_h^X + (n-h)V_\ell^X$.

4.1. **Priority-independent valuations.** In this section, we follow Abdulkadiroğlu et al. (2011) by assuming that the support of f is a finite set $\tilde{\mathcal{V}} \subset \{(v_1, \ldots, v_m) \in [0,1]^m \mid v_1 > v_2 > \cdots > v_m\}$. In particular, all students have the same ordinal preference, preferring school s_a to school s_b if $a < b$, which is conceivable in areas where schools have a clear quality ranking. Importantly, however, students may differ in their relative preference intensities. In this section, we also focus on priority-*in*dependent valuations. That is, every student draws a valuation type $\boldsymbol{v}$ from the *same* distribution $f(\boldsymbol{v})$, regardless of whether the student is a high- or low-priority type (that is, $f_\ell = f_h = f$). The case of priority-dependent valuations is treated in the next section.

Because DA has a truthful dominant strategy, it makes sense to assume that individuals report their preferences truthfully in DA. Thus, because students have the same ordinal preference over schools, every student reports the same ranking of schools in DA. In DA, therefore, the h high priority types are randomly assigned to the h high-quality schools. The ℓ low-priority types are, on the other hand, randomly assigned to the ℓ low-quality schools. Given the (dominant strategy) equilibrium of DA we let $\hat{P}_a^p$ denote the probability that a student of type p is assigned to school s_a, that is

$$\hat{P}_a^h = q_a/h \text{ if } a \in \{1, \ldots, g\}, \text{ and 0 otherwise, and}$$
$$\hat{P}_a^\ell = q_a/\ell \text{ if } a \in \{g+1, \ldots, m\}, \text{ and 0 otherwise.}$$

For a given (symmetric Bayesian) equilibrium σ^* of IA and any strategy σ, let $\dot{P}_a^p(\sigma)$ be the probability that a student with priority type p is assigned to school s_a if that student plays strategy σ and all other students play the symmetric equilibrium strategy σ^*.[15] An equilibrium σ^* of IA is **segregating** if high-priority types are never assigned to a low-quality school. That is, $\sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \dot{P}_y^h(\sigma_h^*(\boldsymbol{v}))f_h(\boldsymbol{v}) = 0$ for every low-quality school $s_y \in \{s_{g+1}, \ldots, s_m\}$ (as in the dominant strategy equilibrium of DA). In contrast, an equilibrium σ^* of IA is **blending** if high-priority types are sometimes assigned to at least one low-quality school. That is, $\sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \dot{P}_y^h(\sigma_h^*(\boldsymbol{v}))f_h(\boldsymbol{v}) > 0$ for some low-quality school $s_y \in \{s_{g+1}, \ldots, s_m\}$.

[14]Finiteness is assumed to simplify the existence of Bayesian equilibria in IA.

[15]A strategy σ is a mapping from the set of valuation types into the set of mixed strategies over reported preferences.

Example 1 (Blending equilibria with IA)**.** Suppose there are three schools, two high-quality schools s_{H1} and s_{H2}, and one low-quality school s_L. The distribution of valuations is degenerate $f(\boldsymbol{v}^*) = 1$ with $v^*_{H1} > v^*_{H2} > v^*_L$. High-priority types can only rank one high-quality school first and there is always a high-quality school H^* that less than q_{H^*} high-priority types rank first. Therefore, low-priority types have a nonzero probability of being assigned to s_{H^*} if they rank s_{H^*} first. As a consequence, low-priority types always secure a payoff strictly larger than $v^*(s_L)$ in equilibrium. This, in turn, implies that the equilibrium of IA must be blending because low-priority types would get a payoff of v^*_L in a segregating equilibrium.

The next theorem shows that low-priority types always prefer the equilibrium outcome of IA to the dominant strategy outcome of DA.

Theorem 2. *For any priority-independent distribution f, any valuation type $\boldsymbol{v} \in \tilde{\mathcal{V}}$, and any symmetric equilibrium of IA, we have $V^{IA}_{\ell,\boldsymbol{v}} \geq V^{DA}_{\ell,\boldsymbol{v}}$, and $V^{IA}_{\ell,\boldsymbol{v}} > V^{DA}_{\ell,\boldsymbol{v}}$ if the equilibrium of IA is blending.*

Proof. Let σ^* be any symmetric equilibrium of IA, with σ^*_ℓ the strategy played by the low-priority types and σ^*_h the strategy played by the high-priority types. For every school s_a, the feasibility constraint (respecting quotas at schools) and the fact that IA is non-wasteful imply

$$\ell \sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \dot{P}^\ell_a(\sigma^*_\ell) f(\boldsymbol{v}) + h \sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \dot{P}^h_a(\sigma^*_h) f(\boldsymbol{v}) = q_a. \tag{2}$$

Equality (2) simply expresses the fact that in an equilibrium of IA, all the seats are distributed and feasibility constraints are respected.

Fix any valuation type $\ddot{v} \in \tilde{\mathcal{V}}$. We show that when student i with valuation type $\ddot{v}$ is a low-priority type, (2) implies that if other students play the equilibrium strategies specified by σ^*, student i can always play a strategy $\tilde{\sigma}$ that makes her at least as well off as under DA. The proposition then follows from the fact that i must be at least as well-of in the equilibrium σ^* as she is when she plays $\ddot{\sigma}$ (and others play according to σ^*).

Let $\ddot{\sigma}_\ell \coloneqq \sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \sigma^*_\ell(\boldsymbol{v}) f(\boldsymbol{v})$. That is, $\ddot{\sigma}_\ell$ involves playing $\sigma^*_\ell(\boldsymbol{v})$ with probability $f(\boldsymbol{v})$, i.e., according to the probability distribution of priority types that play that strategy in the equilibrium σ^*. For any school s_a, the probability that the student is assigned to s_a when she plays $\ddot{\sigma}_\ell$ and others play the equilibrium strategy is

$$\dot{P}^\ell_a(\ddot{\sigma}_\ell) = \dot{P}^\ell_a\left(\sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \sigma^*_\ell(\boldsymbol{v}) f(\boldsymbol{v})\right) = \sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \dot{P}^\ell_a(\sigma^*_\ell(\boldsymbol{v})) f(\boldsymbol{v}) = \frac{q_a - h\sum_{\boldsymbol{v}\in\tilde{\mathcal{V}}} \dot{P}^h_a(\sigma^*_h) f(\boldsymbol{v})}{\ell}, \tag{3}$$

were the last equality follows from (2).

Observe that (3) implies

$$\dot{P}^\ell_y(\ddot{\sigma}_\ell) \leq \hat{P}^\ell_y, \qquad \text{for every low-quality school } s_y \in \{s_{g+1}, \ldots, s_m\}. \tag{4}$$

Because $\hat{P}^\ell_a = 0$ for all $a \in \{g+1, \ldots, m\}$, we also have

$$\dot{P}^\ell_x(\ddot{\sigma}_\ell) \geq \hat{P}^\ell_x, \qquad \text{for every high-quality school } s_x \in \{s_1, \ldots, s_g\}. \tag{5}$$

Specifically, any inequality in (4) is strict if $\sum_{\boldsymbol{v} \in \tilde{\mathcal{V}}} \dot{P}^h_y(\sigma^*_h) f(\boldsymbol{v}) > 0$. That is, one of these inequalities is strict if the equilibrium of IA is blending. As a consequence, if the equilibrium of IA is blending, *some* of the inequalities in (5) must also be strict.

Together, (4) and (5) imply that for any valuation type $\boldsymbol{v} \in \bar{\mathcal{V}}$,

$$V^{IA}_{\ell,\boldsymbol{v}} \geq \sum_{a \in A} v_a \dot{P}^\ell_a(\sigma^*_\ell) \geq \sum_{a \in A} v_a \hat{P}^\ell_a = V^{DA}_{\ell,\boldsymbol{v}},$$

where the last inequality is strict if the equilibrium of IA is blending. □

By the definition of V^X_p, we have the following *ex ante* corollary of Theorem 2.

Corollary 5. *For any priority-independent distribution f and any symmetric equilibrium of IA, we have $V^{IA}_\ell \geq V^{DA}_\ell$, and $V^{IA}_\ell > V^{DA}_\ell$ if the equilibrium of IA is blending.*

Abdulkadiroğlu et al. (2011) show that in the absence of preexisting priorities (that is when, unlike in our model, *all* priorities are determined through tie-breaking), all students are always better off under IA than under DA (as opposed to low-priority types only in Theorem 2). As Troyan (2012) shows, this is not true in the presence of preexisting priorities. In particular, with two priority types as in the present model, some high-priority types can be worse off under IA than under DA (Troyan, 2012, Examples 1 and 2). This is intuitive: In DA, high-priority types are guaranteed an assignment at one of the high-quality schools whereas in IA, low-priority types can "steal" seats at high-quality schools, hence making some high-priority types worse off than in DA.

Troyan (2012) however shows that from an ex ante perspective, IA remains preferable to DA *even* in the presence of preexisting priorities. That is, when one considers a student's expected utility *before* she draws her priority type, the student is better off under IA than under DA. In our context and our terminology, Proposition 2 in Troyan (2012) notably implies that the allocative welfare of IA is greater than the allocative welfare of DA, that is, $V^{IA} \geq V^{DA}$.

Importantly, this result requires us to assume that the distribution of priority is the same for every student. In practice, this is rarely the case. Students' priority types typically correlate with their characteristics such as parents' income, home location, intellectual abilities, etc. This puts in question the use of allocative efficiency as a social objective. If $V^X > V^Y$ but Y favors disadvantaged students, one may have a legitimate preference for Y. The proof of Theorem 2 shows that IA promotes access to high-quality schools for students with low priorities (as was already suggested in a special case by Abdulkadiroğlu et al., 2011, Theorem 3). Suppose that disadvantaged students tend to be the students with low priorities, as can be expected, for example, if priorities follow from test scores. Then Theorem 2 indicates that IA is better than DA in helping disadvantaged students secure a higher-quality school, which complements and reinforces Troyan's comparison in terms of allocative welfare.[16]

Together with most of the school choice literature (including Abdulkadiroğlu et al., 2011), Troyan (2012) also focuses on the welfare effects of mechanism selection mediated by assignments to schools themselves. As we argued, the choice of a mechanism impacts welfare beyond determining students' assignments. In particular, by changing the value associated with different priority types, a change in the allocation mechanism can change the "rent-seeking" behavior that leads to the acquisition of these priority types. Mechanisms that increase the utility wedge between high- and low-priority types may foster fiercer competition for higher priorities, which can increase wasteful effort. The natural question is therefore whether the efficiency advantage of IA over DA is robust to the addition of an effort stage, that is, whether net welfare $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a})$ is also higher than net welfare $W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$). The next proposition answers positively when valuations are priority-independent.

Proposition 4. *For any priority-independent distribution f and any symmetric equilibrium of IA (i) $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) \geq W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$, and (ii) $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) > W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$ if the equilibrium of IA is blending and there is at least one student in the competitive set under DA (i.e., $C(\Delta^{DA}, \boldsymbol{a}) \geq 1$).*

Proof. **(i).** By Corollary 5, $V_\ell^{IA} \geq V_\ell^{DA}$. Thus, if we also have $V_h^{IA} \geq V_h^{DA}$, the proposition follows directly from Proposition 1. Hence, assume that $V_h^{IA} < V_h^{DA}$. By Corollary 1, the difference between the two mechanisms' net welfare is

$$W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) - W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a}) = \underbrace{\left(V^{IA} - V^{DA}\right)}_{:=\Omega_1} + \underbrace{\left(D(\Delta^{DA}, \boldsymbol{a}) - D(\Delta^{IA}, \boldsymbol{a})\right)}_{:=\Omega_2}.$$

By Troyan (2012, Proposition 2), we have $V^{IA} \geq V^{DA}$, which implies $\Omega_1 \geq 0$. Also, together, $V_\ell^{IA} \geq V_\ell^{DA}$ and $V_h^{IA} < V_\ell^{DA}$ imply $\Delta^{DA} > \Delta^{IA}$. By the definition of the deadweight loss term, this implies that the deadweight loss is larger in DA than in IA, which in turn implies $\Omega_2 \geq 0$.

(ii). Because the equilibrium is blending, we have $V_\ell^{IA} > V_\ell^{DA}$ by Corollary 5. Thus, if we also have $V_h^{IA} \geq V_h^{DA}$, the proposition follows directly from Proposition 1. Hence, assume that $V_h^{IA} < V_h^{DA}$ which together with $V_\ell^{IA} > V_\ell^{DA}$ implies $\Delta^{DA} > \Delta^{IA}$. Recall that $\Omega_1, \Omega_2 \geq 0$ by the proof of (i). Therefore, $\Omega_2 > 0$ is sufficient to have the desired result. By definition, the set of competitive students grows with Δ, and the set of competitive students under IA is therefore a subset of the same set under DA. Because only competitive students contribute to the deadweight loss and each of these students' contribution increases with Δ, we therefore have $\Omega_2 > 0$. □

[16] An argument similar to the proof of Theorem 2 can be used to show that *regardless of the number of priority types*, the *lowest* priority type is always better off under IA than under DA. In that sense, *most* disadvantaged students prefer IA to DA even when there are more than two priority types.

Importantly, the ex ante efficiency advantage of IA over DA demonstrated by Troyan (2012, Proposition 2) relies on students having heterogeneous *cardinal* preferences over schools. When heterogeneity in cardinal preferences vanishes, most mechanisms tend to provide the same allocative welfare, and it becomes impossible for one mechanism to dominate another in terms of ex ante (allocative) efficiency.

Observation 1. *If $v_i = v$ for all $i \in \{1, \ldots, m\}$, then for any two non-wasteful mechanisms M and M', the allocative welfare $V^M = V^{M'}$. In particular, in this case, $V^{IA} = V^{DA}$.*

However, even in this extreme case (fully homogeneous cardinal preferences), Proposition 4 shows that IA remains more efficient than DA in terms of net welfare. This is because, even when no allocative efficiency gains are possible, IA still favors low-priority types over high-priority types, which in turns means that wasteful competition is reduced in the effort stage.[17]

Observation 2. *Proposition 4 applies even when $v_i = v$ for all $i \in \{1, \ldots, m\}$ and $V^{IA} = V^{DA}$. In particular, also in this case, $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) > W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$ if the equilibrium of IA is blending and there is at least one student in the competitive set under DA.*

4.2. **Priority-dependent valuations.** In the presence of preexisting priorities, the assumption that valuations are independent of priorities can be problematic. Instead, students' priority types are often correlated with their valuation types. For example, students who live inside a neighborhood usually have higher priorities in the schools in that neighborhood. These students may also have a more intense preference for the schools in that neighborhood – because of shorter commute times or because parents want their children to attend the same school as their neighbors – which results in priority-dependent valuations.

The results in the previous section are based on priority-independent valuations. This is true, for example, for Proposition 4 which relies on Theorem 2 to show that the ex ante efficiency advantage of IA over DA (Troyan, 2012, Proposition 2) is robust to the addition of an effort stage. When correlations are introduced, Theorem 2 does not necessarily apply and the picture becomes more complex.

On the one hand, IA enables efficiency gains by incentivizing students to reveal information about their cardinal preferences through their ordinal reports. Even when valuations correlate with priority types, priorities that result from tie-breaking carry no information on cardinal utility. In an equilibrium of IA, it is possible that some of these priorities are violated in a welfare-improving way (compared to DA).

[17]Formally, even when $V^{IA} = V^{DA}$, we still have $V_\ell^{DA} \le V_\ell^{IA}$ and $V_h^{DA} \ge V_h^{IA}$, which by Corollary 5 implies $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) \ge W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$

On the other hand, IA also enables violations of preexisting priorities which carry useful information on cardinal utility if valuations are priority-dependent.[18] As a consequence, IA may not take full advantage of correlations between priorities and valuations. In contrast, DA respects priorities and therefore takes full advantage of the correlations between preferences and priorities. DA however does not incentivize cardinal preference revelation and therefore misses some efficiency improvement opportunities.

To shed light on these two effects, we generalize the model of the previous section. We now assume that low-priority types and high-priority types draw valuation types from different distributions f_ℓ and f_h, with $f_\ell \neq f_h$. Recall that the priority types on which these distributions depend are *acquired* through the first effort stage. That is, students here develop a particular taste for some schools, as *a result of* the effort they exerted to secure higher priority in these schools.

We also relax the assumption of common ordinal preferences and let the finite set of possible utility values be $\bar{\mathcal{V}} \subset [0,1]^m$, without requiring that $v_1 > \cdots > v_m$ for every $\boldsymbol{v} \in \bar{\mathcal{V}}$. That is, although we keep the distinction between high- and low-quality schools in terminology, we allow students to prefer low-quality schools to high-quality schools (e.g., because they live closer to a low-quality school).

Allowing valuations to be correlated with priority types, it is not hard to find examples of f_ℓ and f_h for which DA is more *allocatively* efficient than IA. For example, suppose that f_ℓ and f_h satisfy the following properties:

a) For all $\boldsymbol{v}, \boldsymbol{v}'$ in the support of (f_h), $v_i = v'_i$ for all $i \in \{1, \ldots, g\}$, and for all $\boldsymbol{v}, \boldsymbol{v}'$ in the support of (f_ℓ), $v_i = v'_i$ for all $i \in \{g+1, \ldots, m\}$.
b) If $\boldsymbol{v}$ is in the support of (f_ℓ) or (f_h), then $v_i > v_j$ for all $i \in \{1, \ldots, g\}$ and all $j \in \{g+1, \ldots, m\}$.
c) For all $\boldsymbol{v}$ in the support of f_h and all $\boldsymbol{v}'$ in the support of f_ℓ, we have $v_i > v'_i$ for all $i \in \{1, \ldots, g\}$

In words, property a) says that cardinal preferences are homogeneous at high-quality schools among high-priority types and at low-quality schools among low-priority types. Property b) says that ordinal preferences between high- and low-quality schools are maintained: Every student still prefers any high-quality school to any low-quality school. Finally, property c) says that the value a high-priority type associates with a high-quality school is always higher than the value a low-priority type associates to that school. If f_ℓ and f_h satisfy a), b), and c), we say that (f_ℓ, f_h) **favors DA**.

If (f_ℓ, f_h) favors DA, then DA is more allocatively efficient than IA. Intuitively, a) reduces the opportunities for efficiency improvements in IA, while c) makes respecting priorities optimal from the point of view of allocative efficiency. Finally, b) guarantees that the (dominant strategy) outcome of DA respects priorities.

[18] In IA, a student from a given district may be assigned with positive probability to a school in another district, even if she has a lower average value at that school than the average value among students in the district of that school.

Observation 3. *If* (f_ℓ, f_h) *favors DA, then* $V^{DA} \geq V^{IA}$ *and* $V^{DA} > V^{IA}$ *if the equilibrium of* IA *is blending.*

Proof. By c), allocating a high-quality school to a high-priority type is always preferable to allocating the same school to a low-priority type from the point of view of allocative efficiency. By a), given that high-quality schools are allocated to high-priority types only, the allocation of these schools among high-priority types is irrelevant from the point of view of allocative efficiency. Similarly, by a), if low-quality schools are allocated to low-priority types only, then the allocation of these schools among low-priority types is irrelevant from the point of view of allocative efficiency.

Thus, any allocation of schools that assigns high-priority types to high-quality schools and low-priority types to low-quality schools exclusively is optimal from the point of view of allocative efficiency. By b), this is true of every allocation in the support of DA given f_ℓ and f_h. Hence V^{DA} is optimal and we have $V^{DA} \geq V^{IA}$. It is also straightforward from the above argument that $V^{DA} > V^{IA}$ if the equilibrium of IA is blending. □

More generally, $V^{DA} > V^{IA}$ if DA's efficiency gains from exploiting correlations between priorities and valuations outweigh IA's efficiency gains from incentivizing cardinal preference revelation, which may occur under conditions on f_ℓ and f_h milder than DA-favorability.

Interestingly, for a class of pairs (f_ℓ, f_h) including DA-favorable pairs, Theorem 2 extends to the more general model studied in this section. That is, for (f_ℓ, f_h) in this class, the low-priority types remain better off in IA than in DA. In turn, this implies that even if $V^{DA} > V^{IA}$, the net welfare of DA remains lower than the net welfare of IA when $\boldsymbol{a}$ is sufficiently competitive.

Consider the following property of f_ℓ:

d) For all $\boldsymbol{v}$ in the support of f_ℓ, we have $v_{g+1} > v_{g+2} > \cdots > v_m$.

In words, property d) says that *ordinal* preferences over low-quality schools are homogeneous *among low-priority types*. If (f_ℓ, f_h) satisfies b) and d), we say that (f_ℓ, f_h) is **sufficiently homogeneous**. Clearly, a) implies d) and (f_ℓ, f_h) being DA-favorable implies that the pair is also sufficiently homogeneous.

The following theorem generalizes Theorem 1 to priority-dependent valuations under the assumption that (f_ℓ, f_h) is sufficiently homogeneous.

Theorem 3. *For any sufficiently homogeneous* (f_ℓ, f_h)*, any valuation type* $\boldsymbol{v} \in \bar{\mathcal{V}}$*, and any symmetric equilibrium of* IA*, we have* $V^{IA}_{\ell,\boldsymbol{v}} \geq V^{DA}_{\ell,\boldsymbol{v}}$*, and* $V^{IA}_{\ell,\boldsymbol{v}} > V^{DA}_{\ell,\boldsymbol{v}}$ *if the equilibrium of* IA *is blending.*

Proof. By b), high-priority types always prefer high-quality schools to low-quality schools. Hence, high-priority types always rank all high-quality schools higher than any low-quality school in the dominant strategy equilibrium of DA. In DA, all the seats at high-quality schools are therefore assigned uniformly at random to high-priority types. By d), the low-priority type all report the same ranking over low-quality schools. Because all seats at high-quality schools are occupied by high-priority types, the ℓ low-priority type are therefore assigned uniformly at random to one of the ℓ low-quality schools. Therefore, the probability that a low-priority type student is assigned to some school s_a under the dominant strategy outcome of DA is again $\hat{P}_a^\ell$.

The rest of the proof is similar to the proof of Theorem 2. For every school s_a, the feasibility constraint (respecting quotas at schools) and the fact that IA is non-wasteful now imply that at any equilibrium σ^* of IA,

$$\ell \sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \dot{P}_a^\ell(\sigma_\ell^*) f_\ell(\boldsymbol{v}) + h \sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \dot{P}_a^h(\sigma_h^*) f_h(\boldsymbol{v}) = q_a, \tag{6}$$

where compared to (2), we have only added indices ℓ and h to the density functions.

The strategy that makes any type in IA at least as well off as in DA is now $\ddot{\sigma}_\ell :=$ $\sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \sigma_\ell^*(\boldsymbol{v}) f_\ell(\boldsymbol{v})$, where again, we only added the superscript "ℓ" to the distribution (compared to the corresponding strategy in the proof of Theorem 2).

We then have

$$\dot{P}_a^\ell(\tilde{\sigma}_\ell) = \dot{P}_a^\ell\left(\sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \sigma_\ell^*(\boldsymbol{v}) f_\ell(\boldsymbol{v})\right) = \sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \dot{P}_a^\ell(\sigma_\ell^*(\boldsymbol{v})) f_\ell(\boldsymbol{v}) = \frac{q_a - h\sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \dot{P}_s^h(\sigma^*) f_h(\boldsymbol{v})}{\ell}, \tag{7}$$

where the last equality follows from (6). Again, (7) implies (4) and (5), with each individual inequality in (4) being strict if $\sum_{\boldsymbol{v}\in\bar{\mathcal{V}}} \dot{P}_y^h(\sigma^*) f_h(\boldsymbol{v}) > 0$. Together, (4), (5), and b) imply that for any valuation type $\boldsymbol{v} \in \bar{\mathcal{V}}$,

$$V_{\ell,\boldsymbol{v}}^{IA} \geq \sum_{a\in A} v_a \dot{P}_a^\ell(\sigma_\ell^*) \geq \sum_{a\in A} v_a \hat{P}_a^\ell = V_{\ell,\boldsymbol{v}}^{DA},$$

where the last inequality is strict if the equilibrium of IA is blending. □

Clearly, by definition of V_p^X, the following is a direct corollary of Theorem 3, and generalizes Corollary 5.

Corollary 6. *For any sufficiently homogeneous (f_ℓ, f_h) and any symmetric equilibrium of IA, we have $V_\ell^{IA} \geq V_\ell^{DA}$, and $V_\ell^{IA} > V_\ell^{DA}$ if the equilibrium of IA is blending.*

Corollary 6 implies that although net welfare may be higher under DA than under IA when valuations are priority-dependent, this can only happen if the allocative welfare of DA is higher than the allocative welfare of IA. *Even* when DA is allocatively more efficient than IA, the net welfare also remains higher under IA than under DA if the effort stage is sufficiently competitive.

Proposition 5. *Suppose that (f_ℓ, f_h) is sufficiently homogeneous. (i) If $V^{IA} \geq V^{DA}$, then $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) \geq W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$. (ii) There exists $\boldsymbol{a}$ sufficiently competitive such that $W(V_\ell^{IA}, V_h^{IA}, \boldsymbol{a}) \geq W(V_\ell^{DA}, V_h^{DA}, \boldsymbol{a})$ even if $V^{DA} > V^{IA}$.*

Proof. **(i).** We have $V^{IA} \geq V^{DA}$ and $V_\ell^{IA} \geq V_\ell^{DA}$ and the proof is therefore identical to the proof of Proposition 4(i). **(ii).** The proof follows directly from Corollary 6 and Proposition 2.

□

5. Conclusion

In this paper, we find that, in a context where students compete for priority in a school choice mechanism, IA may have better welfare properties than DA. Troyan (2012) and Abdulkadiroğlu et al. (2011) also provide cases in which IA provides higher welfare than DA but purely from an allocation perspective with exogenous priorities. Our results extend this qualitative conclusion by showing that IA can be preferable to DA from a *net* welfare perspective, including the cost for students to compete for priorities. As we show, this is true *even* when DA provides a more efficient allocation than IA.

We stress, however, that our results should not been seen an unconditional defense of IA. In particular, we have shown in Section 4.2 that when valuations in the matching stage depend on the priorities acquired through the effort stage, DA can have a better allocative efficiency than IA. For IA's *overall* efficiency to dominate DA's, the effort stage then needs to be sufficiently competitive so that DA's higher inequality (between high- and low-priority types) induces a large wasteful effort with DA that overwhelms its advantage in terms of allocative efficiency.

Overall, our main goal was *not* to provide general conclusions on the relative efficiency of IA and DA. Rather, we have attempted to draw attention to the interactions between the choice of a mechanism and the game through which participants acquire priorities. More generally, we have shown how the choice of a mechanism can have economically significant impacts beyond the problem the mechanism is specifically designed to solve.

To further illustrate how the balance can "tip back" in favor of DA, we conclude by discussing another situation in which interactions between the school choice mechanism and the effort stage may results in DA being more efficient than IA.

Throughout this paper, we have assumed that students do *not* know the realization of their utility value for schools at the time they exert effort to secure higher priorities. Instead, we have assumed that prior to the matching stage, students only know the *distribution* from which they will draw values for schools in the matching stage, but they do *not* know *the realization* of this distribution.[19]

[19] Signaling valuation through effort is discussed to some extent in Section 4.2. However, in Section 4.2, students do not differ in their potential valuations between schools *at the beginning of the effort stage*. Students rather *acquire* different valuations over schools as they gain different priorities over these schools (e.g., students acquire a higher preference for schools in a given neighborhood by moving to that neighborhood, which also gives them a higher priority at the schools in that neighborhood). Here, in comparison, we discuss a variant of our model in which students are *initially* heterogeneous in their valuation over schools (that is, heterogeneous from the very beginning of the effort stage) and are able to signal their heterogeneous valuation through heterogeneous efforts.

Suppose instead that students have intrinsically heterogeneous utility over schools, and these utilities are known at the beginning of the effort stage. Students can then signal their heterogeneous valuation directly through the effort stage. This can potentially reverse the efficiency comparison between DA and IA. To understand why, note that if students with a more intense preference for higher-quality schools exert more effort and obtain higher priorities as a consequence, priorities become a signal of preference intensity. In this case, respecting priorities may become desirable from the point of view of efficiency. Because DA is better at respecting priorities than IA, DA could therefore be more efficient than IA.[20]

An example where signaling of intrinsic valuation through effort makes DA more efficient than IA is presented in Appendix Section A.2. As the example illustrates, these efficiency gains can materialize *even when effort types are perfectly competitive* with $a_i = a$ for all $i \in \{1, \ldots, n\}$, which contrasts with Corollary 5. Again, this example points at the importance of better understanding the interaction between the choice of mechanism and the game through which students acquire priorities at schools. More generally, it stresses the value of considering mechanism selection in "general equilibrium" settings, rather than focusing exclusively on the partial equilibrium effect a mechanism has on the allocation problem it is designed to solve.

[20]Although this is different from the kind of signaling discussed in Section 4.2 (see footnote 19), the intuition is similar to that presented in Section 4.2.

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Appendix A. Appendix

A.1. **Stage 1 Contest is *Generic*.** Let $\tilde{u}_i(t) = u_i(t) - V_\ell$. Rewrite $\tilde{u}_i(t) = H_i(t)\, v_i(t_i) - (1 - H_i(t))\, c_i(t_i)$ where $v_i(t_i)$ and $c_i(t_i)$ are defined follows:

$$v_i(t_i) := V_h - V_\ell - (e(t_i) - e(a_i))$$

$$c_i(t_i) := (e(t_i) - e(a_i))$$

$\tilde{u}_i(t)$ is a normalization of u_i with the same constant V_ℓ subtracted from the utility of each player. With this normalization, the contest meets $A1 - A3$ as well as the *power* and *cost* conditions of Siegel (2009).

(A1) v_i and $-c_i$ are continuous since e is assumed to be continuous. v_i and $-c_i$ are strictly decreasing since e is assumed to be strictly increasing.

(A2) $v_i(a_i) = V_h - V_\ell > 0$ by assumption. $\ell im_{s_i \to \infty} v_i(s_i) < c_i(a_i) = 0$ since e is strictly increasing and there exists some $\tilde{t}$ such that $V_h - V_\ell < e(\tilde{t}) - e(a_1)$.

(A3) $c_i(t_i) > 0$ if $v_i(t_i) = 0$ since $v_i(t_i) = V_h - V_\ell - c_i(t_i)$ and $V_h - V_\ell > 0$.

Let the *reach* of a player ρ_i, be the score such that $v_i(\rho_i) = 0$. The reach of a player is the ρ_i that solves $e(\rho_i) = V_h - V_\ell + e(a_i)$. Since the effort function e is identical for each player and is assumed to be strictly increasing, and since the players are indexed by a_i, $\rho_i \geq \rho_j$ for $i < j$. The *threshold* is given by $\tilde{t} = \rho_{h+1}$. A player's *power* is $v_i(\tilde{t}) = V_h - V_\ell - \left(e(\tilde{t}) - e(a_i)\right)$.

(Power Condition) $v_{h+1}(\tilde{t}) = v_{h+1}(r_{h+1}) = 0$ by construction and by the assumption that $a_{h+1} \neq a_j$ for $j \neq h+1$, $v_i(\tilde{t}) \neq 0$ for all $i \neq h+1$.

(Cost Condition) $v_i(t_i)$ is strictly decreasing for $h+1$ at $\tilde{t}$ since e is assumed to be strictly increasing.

A.2. **Signaling differences in intrinsic valuation through effort.** Consider a variation of our model where students have different valuation types $f_i(\boldsymbol{v})$. That is, students now draw valuations over schools from idiosyncratic distributions *which they know at the beginning of the effort stage*. This is unlike the model presented in the paper where *all* high- or low-priority types draw a valuation from the same distribution f_ℓ or f_h, respectively. In that case, valuation types are "acquired" through priorities. Here, we assume that valuation types are independent of priority types and may differ from one student to another. In particular, unlike in Section 4.2, it is in principle possible for a high valuation type to have a low-priority type, and vice versa.

In this case, we want to show that DA can be more efficient than IA even in a perfectly competitive environment, and even when DA induces more wasteful effort. The reason is that in this new setting, the effort becomes a *signal* of students' heterogeneous valuations f_ℓ or f_h. Therefore, even if effort is costly and DA forces students to exert more effort than IA, this can be more than compensated for by the usefulness of the signal from an efficiency standpoint.

To rule out efficiency results driven by insufficient competitiveness, we assume $a_i = a$ for all $i \in \{1, \ldots, n\}$. For simplicity, we also assume a *linear* cost of effort. There are four students and three schools, s_{H_1}, s_{H_2}, and s_L. Schools s_{H_1} and s_{H_2} have one seat, and school s_L has two seats. We consider the case of two valuation types h and ℓ with degenerate valuation distributions f_h and f_ℓ with $f_h(\boldsymbol{v}^h) = 1$ and $f_\ell(\boldsymbol{v}^\ell) = 1$, where $\boldsymbol{v}^h$ and $\boldsymbol{v}^\ell$ are defined as follows:

Schools	H_1	H_2	L
$\boldsymbol{v}^h$	.8	.2	0
$\boldsymbol{v}^\ell$	.55	.25	.20

There are two high valuation types that draw values from f_h, and two low-valuation types that draw values from f_ℓ. Students still compete for high priority in high-quality schools H and HM through effort, with the two highest-effort students getting high priorities in high-quality schools (ties are again broken symmetrically at random).

Let $V_{i,j}^X$ denote the value for allocative welfare of valuation type i and priority type j when the mechanism is X. For example, $V_{h,\ell}^{IA}$ is the value for a high valuation / low priority type when the mechanism is IA. We now compute valuations depending on whether types are aligned, with high valuation types having high-priority types, or misaligned, with high valuation types having low-priority types.

IA **with aligned types.** Observe that ranking school s_L anywhere but last is dominated for all students. Thus, students must choose between the strategy Q_1 consisting of ranking s_{H1} first followed by s_{H2}, and the strategy Q_2 consisting of ranking s_{H2} first followed by s_{H1} (or a mix of these two strategies).

For high valuation types, Q_1 is a dominant strategy.[21] The best response for low valuation types is Q_2, which produces the following valuations:

- $V_{h,h}^{IA} = (1/2) * 0.8 + (1/2) * 0 = 0.4$, and
- $V_{\ell,\ell}^{IA} = (1/2) * 0.25 + (1/2) * 0.20 = 0.225$.

IA **with misaligned types.** Again, ranking school s_L anywhere but last is dominated for all students, and the strategies of the students are Q_1 and Q_2 (or a mix thereof). For

[21] Strategy Q_1 is a strict best-response both when the other high valuation type reports Q_1 and when the other high valuation type reports Q_2.

low valuation types, Q_1 is a dominant strategy.[22] The best response for low valuation types is Q_2, which produces the following valuations:

- $V^{IA}_{h,\ell} = (1/2) * 0.2 + (1/2) * 0 = 0.1$, and
- $V^{IA}_{\ell,h} = (1/2) * 0.55 + (1/2) * 0.20 = 0.375$.

DA **with aligned types.** Valuations are:

- $V^{DA}_{h,h} = (1/2) * 0.8 + (1/2) * 0.2 = 0.5$, and
- $V^{DA}_{\ell,\ell} = 0.20$.

DA **with misaligned types.** Valuations are:

- $V^{DA}_{h,\ell} = 0$, and
- $V^{DA}_{\ell,h} = (1/2) * 0.6 + (1/2) * 0.20 = 0.4$.

This implies

$$V^{IA}_{h,h} - V^{IA}_{h,\ell} = 0.3 > 0.175 = V^{IA}_{\ell,h} - V^{IA}_{\ell,\ell},$$

and

$$V^{DA}_{h,h} - V^{DA}_{h,\ell} = 0.5 > 0.2 = V^{DA}_{\ell,h} - V^{DA}_{\ell,\ell}.$$

Thus, in an equilibrium of the effort stage with linear effort functions, the high valuation types exert the most effort and become high-priority types. Specifically, high valuation types exert 0.175 effort when the mechanism is IA and 0.275 when the mechanism is DA (low valuation types exert no effort).

Therefore, the net welfare under each mechanism is:

- $W(V^{IA}_{\ell}, V^{IA}_{h}, \boldsymbol{a}) = 2 * [0.4 - 0.175] + 2 * 0.225 = 0.9$, and
- $W(V^{DA}_{\ell}, V^{DA}_{h}, \boldsymbol{a}) = 2 * [0.5 - 0.2] + 2 * 0.2 = 1$.

That is, the higher effort cost inherent with DA is more than compensated by the fact that DA makes better use of the valuation type signaling contained in that effort. Of course, this is just one example and other valuation profiles can be found for which the additional cost of effort in DA outweighs DA's better use of effort signaling. This again points to the importance of better understanding the mechanisms through which students acquire priorities and their interactions with allocation mechanisms, and we leave potential general results in the case of intrinsic valuation heterogeneity for future research.

[22] Again, strategy Q_1 is a strict best-response both when the other low valuation type reports Q_1 (yielding 0.3 as a payoff, instead of 0.25 under Q_2) and when the other low valuation type reports Q_2.