A Generalization of Teicher's Poisson Tail Inequality

Greg Leo*

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Abstract

Teicher [1955] proved that the probability a Poisson distribution with mean k takes on a value of k or less is monotonically decreasing in k. I extend this inequality by proving that the probability a Poisson distribution with mean zk takes on a value of k or less is monotonically decreasing for $z \geq 1$.

Let X_λ be a Poisson distributed random variable with mean λ. This note concerns the monotonicity of $P(X_{zk} \leq k)$ in k. Teicher [1955] proves the following [see also: Adell and Jodra, 2005]:

$$P(X_k \leq k) > P\left(X_{(k+1)} \leq k+1\right) \tag{1}$$

Proposition. *$P(X_{zk} \leq k) > P\left(X_{z(k+1)} \leq k\right)$ for $z \geq 1$.*

Proof. Let $G_k(z) \equiv P(X_{zk} \leq k) - P\left(X_{z(k+1)} \leq k\right)$. The proposition is equivalent to $G_k(z) > 0$ for $z \geq 1$. Note that $G_k(1)$ by Adell, Jodra (2005).

*Leo: Vanderbilt University Department of Economics. Station B #351819 2301 Vanderbilt Place Nashville, TN 37235-1819. I thank Pedro Sant'anna for helpful comments. g.leo@vanderbilt.edu.

Furthermore, $lim_{z\to\infty}G_k(z) = 0$. Note that $G_k(z)$ can be rewritten by the following steps:

$$G_k(z) = P(X_{zk} \le k) - P\left(X_{z(k+1)} \le k+1\right) = \\ P(X_{zk} \le k+1) - P\left(X_{z(k+1)} \le k+1\right) - P(X_{zk} = k+1) \tag{2}$$

By the Gamma-Poission relationship, the lower tail of X_λ can be written as the upper-tail of a random variable with distribution $\Gamma(k+1, 1)$. Thus, $P(X_\lambda \le k) = P(Y > \lambda) = \frac{\int_\lambda^\infty u^k e^{-u} du}{k!}$. Using this relationship, 2 can be written:

$$G_k(z) = \frac{\int_{zk}^\infty u^{(k+1)} e^{-u} du}{(k+1)!} - \frac{\int_{z(k+1)}^\infty u^{(k+1)} e^{-u} du}{(k+1)!} - \frac{(zk)^{k+1} e^{-(zk)}}{(k+1)!} \tag{3}$$

The first two terms have the same integrand but different bounds. The expression can be re-written as:

$$G_k(z) = \frac{\int_{zk}^{z(k+1)} u^{(k+1)} e^{-u} du}{(k+1)!} - \frac{(zk)^{k+1} e^{-(zk)}}{(k+1)!} \tag{4}$$

The derivative of $(k+1)!G_k(z)$ with respect to z is:

$$(k+1)!G_k'(z) = (k+1)(z(k+1))^{(k+1)} e^{-z(k+1)} - (k)(zk)^{k+1} e^{-zk} \\ - k(k+1)(zk)^k e^{-zk} + k(zk)^{k+1} e^{-zk} \tag{5}$$

$$= (k+1)(z(k+1))^{(k+1)} e^{-z(k+1)} - k(k+1)(zk)^k e^{-zk} \tag{6}$$

Thus $G_k'(z) > 0$ iff:

$$\left(\frac{k+1}{k}\right)^{(k+1)} > \frac{1}{z} e^z \tag{7}$$

For $z = 1$ this is $\left(\frac{k+1}{k}\right)^{(k+1)} > e$ which is true for all $k > 0$. Thus, $G_k^{'}(1) > 0$. This together with 1 imply that $G_k^{'}(z)$ is initially positive and increasing on $z \in [1, \infty)$. Furthermore, there is a single stationary point since the equation $\left(\frac{k+1}{k}\right)^{(k+1)} = \frac{1}{z}e^z$ has only one solution for $z \geq 1$ which is given by the lower branch of the Lambert-W function:

$$z^* = -W\left(-\left(\frac{k}{k+1}\right)^{(k+1)}\right) \tag{8}$$

Thus, $G_k(z)$ is strictly positive at $z = 1$, increases for $z \leq z^*$ and then decreases for $z \geq z^*$ approaching 0. This implies that $G_k(z) > 0$ for $z \geq 1$.

□

References

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