Volunteer's Dilemmas with and without coordination

Ted Bergstrom and Greg Leo

October 26, 2020

> "... every one is more negligent of what another is to see to, as well as himself, than of his own private business; as in a family one is often worse served by many servants than by a few." *Aristotle's Politics* Book II, Chapter 3

For many tasks, the efforts of a single individual are sufficient to serve an entire group. One sentinel can alert an entire community. One group member can investigate and report on the honesty of a vendor or the usefulness of a consumer good. Thus we might expect clustering into groups to be advantageous.

But as the epigraph from Aristotle suggests, there is a countervailing force. The returns to scale offered by larger groups can be offset by free-rider problems that increase with group size. In this passage, Aristotle makes two assertions, the first claim is that as group size increases, each member is less likely to act in the public interest. The second is the stronger claim that *total* effort diminishes with group size.

1 Volunteer's dilemma with identical players

The forces that underlie Aristotle's claims are nicely captured in a simple n-player game that was introduced by the sociologist, Andreas Diekmann, who named this game, the *Volunteer's Dilemma.* [2] In Diekmann's Volunteer's Dilemma game, as originally presented, players have identical payoff functions and move simultaneously. Each player has the option of taking a costly action *help* or of choosing *not help.* If at least one player chooses *help*, then all players receive benefits b. Those who choose not to help have net benefits b, while those who choose to help have net benefits $b - c > 0$. If no player helps, then all get a payoff of 0.

In this game, with two or more players, there can not be a Nash equilibrium where everyone helps. If everyone else helps, a player is better off not helping than helping. Nor can there be an equilibrium where no one helps. If no one else helps, a player is better off helping than not.

Diekmann shows that for a group of size n, there is a unique symmetric mixed-strategy Nash equilibrium with the following properties.

Proposition 1 (Diekmann)**.** *In a symmetric Nash equilibrium for the Volunteer's Dilemma, as the number of players increases:*

A) the expected utility of each player remains constant and equal to $b - c$.

B) the probability that each player volunteers diminishes.

C) the probability that no player volunteers increases.

D) as the number of players approaches infinity, the probability that no player volunteers asymptotically approaches $\frac{c}{b}$ from below.

Proof. (Algebraic Proof)

In a symmetric equilibrium, let q be the probability that a player does not help. For any player, the probability that somebody else helps is then $1 - q^{n-1}$. A player will be indifferent between helping and not helping if

$$b\left(1 - q^{n-1}\right) = b - c. \tag{1}$$

Equation 1 implies that the probability that any single player does not help is

$$q = \left(\frac{c}{b}\right)^{\frac{1}{n-1}}. \tag{2}$$

Therefore the probability that nobody helps is

$$q^n = \left(\frac{c}{b}\right)^{\frac{n}{n-1}}. \tag{3}$$

Results A,B,C, and D, follow easily from Equations 2 and 3. □

Results A, B, and C can also be proved by a verbal argument that helps to build understanding.

Proof. Verbal Proof

In a symmetric Nash equilibrium, each player is indifferent between helping and not helping. Regardless of the number of players, the expected payoff from helping is simply $b - c$. Therefore the expected payoff to every player in a mixed strategy Nash equilibrium must be $b - c$, which does not change with the number of players. This proves Result A.

The expected utility from not helping is equal to b times the probability that somebody else helps. Therefore the equilibrium probability that somebody else helps must be constant as the number of players increases. If the number of players increases and the probability that someone else helps is constant, it must be that each player is less likely to help. This proves Result B.

Let Player i be any single player. The probability that somebody helps is the probability that somebody other than i helps, plus the probability that i helps and nobody else does. According to Result A, as the number of players increases, the probability that somebody else helps is constant. According to Claim B, the probability that i helps must decrease with the number of players. It follows that the probability that somebody helps must decrease. This proves Result C. □

Diekmann's Result B echoes Aristotle's assertion that "everyone is more negligent of what another is to see to, as well as himself. . . ." Result C affirms Aristotle's stronger claim that "one is often worse served by many servants than a few." Aristotle's proclamations concerned the well-being of the master, but not that of the servants. Result A of this proposition relates to the well-being of the servants. In Diekmann's model, although servants would like to see the job done, they are no worse off as their number increases, because each of them is less likely to have to do the job himself.

2 Volunteer's Dilemma with differing costs and incomplete information

It is common practice to "simplify" game theoretic models by assuming that players have identical payoff functions and to study symmetric mixed-strategy Nash equilibria of the resulting game. It is sometimes believed that this simplifying assumption does not seriously affect qualitative predictions of the model. In the case of Volunteer's Dilemma, allowing players to have differing payoff functions leads to qualitative results quite different from those in Proposition 1.

We consider a model in which payoffs differ and where there is incomplete information. Players know their own values of c and b, but do not know the costs and benefits of the other players. They view the others benefits and costs as independent random draws from a distribution that is common to all. The assumption of incomplete information seems appropriate for games in which players are thrown together by chance for a single interaction.[1]

Assumption 1 (Payoff distribution)**.** *Each player i knows his or her own costs c_i and benefits b_i, but this information is private. All players believe the $n-1$ other players have preferences parameters c and b such that $b > 0$ and the ratios c/b are independent random draws from a distribution with continuous, strictly increasing cumulative density function $F(\cdot)$ with support $[\ell, u]$, where $F(\ell) = 0$, $F(u) = 1$, and where $\ell < 1$ and $u > 0$.*

The assumption that $\ell < 1$ implies that with positive probability, a randomly chosen individual will have $c < b$ and hence would be willing to do the task if he believed that nobody else will do it. This assumption allows the possibility that $\ell < 0$, in which case some players may prefer to help even if they knew that someone else was helping. The assumption that $u > 0$ is requires that there is positive probability that a player has positive costs.

This game can be modelled as a symmetric game, in which a strategy is a function that maps a player's c and benefit b into an action, *Help* or *Not Help*. We define a strategy with a threshold cost-benefit cost ratio as follows.

Definition 1 (Threshold strategy)**.** *In a threshold strategy with cost-benefit threshold ρ, a player with costs c and benefit b plays* Help *if $c/b < \rho$ and plays* Not Help *if $c/b > \rho$.*

Where $F(\cdot)$ is the distribution function of cost-benefit ratios, let $G(\rho) = 1 - F(\rho)$. If all n players use a threshold strategy with threshold ρ, each player will believe that the probability that nobody else will help is $G(\rho)^{n-1}$. Then the expected payoff to a player with benefit b and cost c from the strategy *Not Help* will be $b\left(1 - G(\rho(n)^{n-1}\right)$, while the expected payoff from the strategy *Help* will be $b - c$. If all players use a threshold strategy with cost-benefit threshold ρ, then a player with cost c and benefit b will be indifferent between helping and not helping if

$$b\left(1 - G(\rho)^{n-1}\right) = b - c, \tag{4}$$

or equivalently,

$$G(\rho)^{n-1} = c/b = \rho. \tag{5}$$

[1]Situations where the same players are engaged in repeated encounters and know each other well might better be treated as games of complete information. Weesie[3] characterizes asymmetric equilibria for Volunteer's Dilemma games with differing payoffs, but with complete information in which players know each other's payoffs.

Proposition 2 asserts that for any integer n, Equation 5 will have a unique solution $\rho(n)$, When all players use the threshold strategy with threshold $\rho(n)$, those players for whom $c/b < \rho(n)$ will choose the strategy *Help* and those for whom $c/b > \rho(n)$ will choose *Not Help.* Therefore the threshold strategy with threshold $\rho(n)$ is the unique Nash equilibrium threshold strategy. This equilibrium is described by the following proposition, which is proved in Appendix A1.

Proposition 2. *In an n-player Volunteer's Dilemma game with payoff distribution satisfying Assumption 1:*

A) for all positive integers, n, there is a unique Nash equilibrium threshold strategy with threshold cost-benefit ratio $\rho(n) \in [\ell, b]$.

B) The threshold ratio $\rho(n)$ decreases as n increases.

C) $\lim_{n\to\infty} \rho(n) = \max\{0, \ell\}$.

Claim B of Proposition 2 shows that the first part of Aristotle's pronouncement still applies. The larger the number of players, the less likely it is that any one of them will help. But what about Aristotle's stronger claim that "a family is often worse served by many servants than a few"? Note that Aristotle qualified this claim by saying "often", not "always". In the Volunteer's dilemma game with differing preferences, we see that Aristotle's qualified claim is accurate. The probability that in equilibrium nobody helps may either increase or decrease with the number of players, depending on the shape of the distribution F of cost-benefit ratios.

The effect of the number of players on the probability that nobody helps depends on the *"elasticity of refusals"*, which we define as the elasticity of the function G with respect to ρ. We say that refusals are *elastic* or *inelastic* with respect to cost-benefit ratio, depending on whether the elasticity of refusals is smaller than or greater than -1. More formally:

Definition 2 (elasticity of refusals)**.** *The elasticity of refusals $\eta_r(\rho)$ at cost-benefit ratio ρ is given by*

$$\eta_r(\rho) = \frac{\rho G'(\rho)}{G(\rho)}.$$

Refusals are said to be cost-benefit elastic *if $\eta_r(\rho) < -1$ and* cost-benefit inelastic *if $-1 < \eta_r(\rho) < 0$.*

We have the following result.

Proposition 3. *In a volunteer's dilemma where the distribution of costs and benefits satisfies Assumption 1, the equilibrium probability that no one helps increases with the number of players if refusals are cost-benefit inelastic and decreases with the number of players if refusals are cost-benefit elastic.*

Proof. Let $\rho(n)$ be a Nash equilibrium threshold for n players. Then $G(\rho(n))^n$ is the equilibrium probability that no one helps. In a threshold equilibrium with n players, a player for whom $\rho = \rho(n)$ must be indifferent between helping and not helping. From Equation 5 it follows that $G(\rho(n))^{n-1} = \rho(n)$. Therefore

$$G(\rho(n))^n = \rho(n)G(\rho(n)). \tag{6}$$

From Equation 6 it follows that the probability that no one helps increases or decreases with n depending on whether $\rho(n)G(\rho(n))$ decreases or increases with n. According to Proposition 2, $\rho(n)$ decreases as n increases. Therefore an increase in n will increase the probability that no one helps if $\rho G(\rho)$ is a decreasing function of ρ. Straightforward calculus shows that $\rho G(\rho)$ is a decreasing (increasing) function of ρ if refusals are cost-elastic (cost-inelastic) for all $\rho \in [\ell, u]$. □

An example: The Pareto distribution

The Pareto distribution has a constant elasticity of refusals over the entire domain of cost-benefit ratios. Since this elasticity can be chosen to be either greater or less than -1, we can use the Pareto distribution to construct examples where the probability that nobody helps either increases or decreases with the number of players.

The Pareto distribution functions used in our example take the form

$$F(\rho) = 1 - \ell^{\alpha}\rho^{-\alpha} \tag{7}$$

with $\alpha > 0$ and with support $[\ell, \infty)$ where $0 < \ell < 1$. Then

$$G(\rho) = 1 - F(\rho) = \ell^{\alpha}\rho^{-\alpha}. \tag{8}$$

Then for all $\rho \in [\ell, \infty)$, the elasticity of refusals is $\frac{\rho G'(\rho)}{G(\rho)} = -\alpha$.

With n players, a Nash equilibrium threshold $\rho(n)$ must satisfy Equation 5. For the Pareto distribution, this implies that

$$\ell^{\alpha(n-1)}\rho(n)^{-\alpha(n-1)} = \rho(n), \tag{9}$$

which implies that

$$\rho(n) = \ell^{\frac{\alpha(n-1)}{1+\alpha(n-1)}}. \tag{10}$$

Substituting from Equation 10 into Equation 8, we find that

$$G\left(\rho(n)\right) = \ell^{\frac{\alpha}{1+\alpha(n-1)}}. \tag{11}$$

Then the probability that in equilibrium with n players, nobody helps is

$$G(\rho(n))^n = \ell^{\frac{\alpha n}{1+\alpha(n-1)}}. \tag{12}$$

From Equation 12, we see that the limit as n approaches infinity, the probability that nobody contributes approaches ℓ. We can also verify that if $0 < \alpha < 1$, this probability is decreasing in n and approaches ℓ from above. If $\alpha = 1$, this probability is constant, and if $\alpha > 1$ this probability is increasing in n and asymptotically approaches ℓ from below.

3 Coordinated Volunteer's Dilemmas

In Diekmann's Volunteer's Dilemma, everyone who offers to help must bear the cost of helping, even though only one player's help is needed. Sometimes the efforts of volunteers can be managed more efficiently. An organization may be able to solicit volunteers for a one-person task and then choose just one volunteer who is asked to perform the task. For example, the *National Marrow Donor Program* (NMDP) maintains a registry of persons who avow their willingness to donate stem cells to a leukemia patient with a matching immune system if the need should occur. For patients with common immunity types, there are likely to be many eligible volunteers in the registry. The NMDP chooses just one of the matching registrants to make the donation.[1]

We define a Coordinated Volunteer's Dilemma as follows:

Definition 3 (Coordinated Volunteer's Dilemma)**.** *In an n-player Coordinated Volunteer's Dilemma, there is a task to be performed. Each player has two possible pure strategies:* Volunteer *and* Refuse*. If no player volunteers, the task is not done and all players receive a payoff of 0. If one or more players volunteer, exactly one volunteer is randomly selected to perform the task. In this case, each player i receives a benefit b_i. If player j is selected to do the task, then player j must pay a cost c_j and thus receives a net benefit of $b_j - c_j$.*

3.1 Coordinated Volunteer's Dilemma with identical payoffs

Suppose that all players gain the same benefits b from having the task performed and all have the same cost c for performing it. In this case there cannot be a pure strategy Nash equilibrium in which everyone volunteers, nor can there be a pure strategy Nash equilibrium in which no one volunteers. There will, however, be a unique symmetric mixed strategy Nash equilibrium in which each player volunteers with probability $p = 1 - q$ and refuses with probability q. As we show in Appendix A.1, when $0 < \rho < 1$, the equilibrium value $q(\rho, n)$ is the unique solution to the equation

$$H(q(\rho, n), n) = \rho \tag{13}$$

where we define

$$H(q, n) = n \left(\frac{q^n}{1 - q^n} \right) \left(\frac{1 - q}{q} \right) \tag{14}$$

Proposition 4, which is proved in Appendix A.1, informs us that Aristotle's remark "every one is more negligent of what another is to see to." applies to coordinated as well as uncoordinated Volunteer's Dilemmas.

Proposition 4. *In the Coordinated Volunteer's dilemma game with $n \geq 2$ identical players, and with $b > c$, there is a unique symmetric mixed strategy Nash equilibrium. The equilibrium probability $q(\rho, n)$ that an individual refuses to volunteer is increasing in the cost-benefit ratio ρ and is also increasing in the number of players.*

It is not surprising that as the number of players increases, the probability that any one of them volunteers diminishes. More surprisingly, even with coordination of volunteers, the symmetric equilibrium probability that nobody volunteers increases with the number of players. Thus Aristotle's dictum that "one is often worse served by many servants than by a few" applies in strong form, with the word "often" replaced by "always".

Definition 4. *Where $q(\rho, n)$ is the equilibrium probability that any individual refuses to help when cost-benefit ratios are equal to ρ and the number of players is n, define the probability $Q(\rho, n) = q(\rho, n)^n$ to be the probability that nobody contributes.*

We have the following result which is proved in Appendix A.2.

Proposition 5. *In the symmetric Nash equilibrium of the Coordinated Volunteer's Dilemma game with identical players, the larger the number of players, the greater is the probability that nobody volunteers.*

Numerical Solutions

Tables 1-3 show numerical solutions for the equilibrium probabilities that individuals do not volunteer and that noone volunteers as the number of players vary.[2] In these tables, p is the probability that each player volunteers, $1-q^n$ is the probability that some player volunteers, and $E(u) = b\left(1 - q^{n-1}\right)$ is the expected utility of each player in symmetric Nash equilibrium.

Table 1: Probabilities of Volunteering and Expected Utility in Uncoordinated and Coordinated Volunteer's Dilemmas with $c/b = 0.9$

Number of Players	Uncoordinated			Coordinated		
	p	$1-q^n$	E(u)	p	$1-q^n$	E(u)
2	0.100	0.190	0.10b	0.182	0.331	0.1818b
3	0.051	0.146	0.10b	0.097	0 .263	0.1844b
4	0.035	0.131	0.10b	0.066	0.231	0.1853b
5	0.026	0.123	0.10b	0.050	0.227	0.1857b
50	0.002	0.001	0.10b	0.004	0.190	0.1864b
100	0.001	0.102	0.10b	0.002	0.189	0.1870b
200	0.0005	0.100	0.10b	0.001	0.188	0.1871b

[2] Calculations were done with MatLab's polynomial solver. For this purpose it is handy to rearrange the equilibrium condition

$$bq^{n-1} = c\frac{1-q^n}{n(1-q)}$$

in the form

$$(n\frac{b}{c} - 1)q^n - n\frac{b}{c}q^{n-1} + 1 = 0.$$

Table 2: Probabilities of Volunteering and Expected Utility in Uncoordinated and Coordinated Volunteer's Dilemmas with $c/b = .5$

Number of	Uncoordinated			Coordinated		
Players	p	$1-q^n$	E(u)	p	$1-q^n$	E(u)
2	0.500	0.250	0.5b	0.667	0.889	0.6667b
3	0.293	0.646	0.5b	0.442	0.826	0.6883b
4	0.206	0.603	0.5b	0.388	0.796	0.6967b
5	0.159	0.580	0.5b	0.261	0.779	0.7011b
50	0.014	0.507	0.5b	.974	0.721	0.7137b
100	0.007	0.503	0.5b	0.013	0.718	0.7147b
200	0.003	0.502	0.5b	0.006	0.717	0.7150b

Table 3: Probabilities of Volunteering and Expected Utility in Uncoordinated and Coordinated Volunteer's Dilemmas with $c/b = .25$

Number of	Uncoordinated			Coordinated		
Players	p	$1-q^n$	E(u)	p	$1-q^n$	E(u)
2	0.750	0.938	0.75b	0.857	0.980	0.8571b
3	0.500	0.875	0.75b	0.650	0.957	0.8772b
4	0.370	0.843	.75b	0.514	0.944	0.8852b
5	0.293	0.823	0.75b	0.423	0.936	0.8894b
50	0.028	0.757	0.75b	0.046	0.906	0.9021b
100	0.014	0.753	0.75b	0.023	0.905	0.9028b
200	0.007	0.752	0.75b	0.007	0.904	0.9031b

Limiting Values

According to Proposition 5 for any $\rho \in (0,1)$, the sequence of equilibrium probabilities $Q(\rho, n)$ of probabilities of that there will be no volunteers is bounded and increasing in n, and hence must approach a limit. This permits the following definition.

Definition 5. *For all* $\rho \in (0,1)$*, define* $\bar{Q}(\rho) = \lim_{n\to\infty} Q(\rho, n)$.

The function $\bar{Q}(\rho)$ can not in general be expressed in terms of standard elementary functions. However, as we will show, $\bar{Q}(\rho)$ is the inverse of a function that is of simple closed form. This function, with domain $(0,1)$ is defined by

$$\bar{\rho}(Q) = \frac{-Q}{1-Q} \ln Q. \tag{15}$$

As we demonstrate in the Appendix, $\rho(Q)$ is a strictly increasing function from $(0,1)$ onto itself and hence its inverse $\bar{\rho}^{-1}(\rho)$ is also a well-defined, increasing function from $(0,1)$ onto itself such that $\bar{\rho}^{-1}(\rho) = \bar{Q}(\rho)$. A graph of this function is shown in Figure 1.

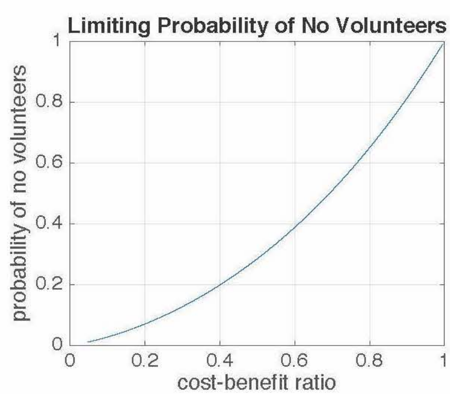

Figure 1: Graph of $Q(\rho)$

Proposition 6. *As the number of players approaches infinity, the limiting probability that nobody volunteers when the cost-benefit ration is* ρ *is given by a function* $\bar{Q}(\rho)$ *that is strictly increasing on* $(0,1)$ *and is the inverse of* $\bar{\rho}(Q) = \frac{-Q}{1-Q} \ln Q$.

3.2 Coordinated Volunteer's Dilemma with differing costs and incomplete information

Suppose that cost-benefit ratios differ among players as described in Assumption 1, with players knowing their own cost-benefit ratios and having common priors about the distribution of cost-benefit ratios of others. As in the uncoordinated game, this can be modelled as a symmetric game in which a strategy maps one's own cost-benefit ratio to an action, *Volunteer* or *Not Volunteer*. A symmetric equilibrium takes the form of a threshold cost benefit ratio $\rho(n)$ such that players with cost-benefit ration $c/b < \rho(n)$ volunteer and those for whom $c/b > \rho(n)$ do not volunteer.

Suppose that players' cost-benefit ratios are independently drawn from a distribution function $F(\rho)$ and let $G(\rho) = 1 - F(\rho)$. If there is a symmetric Nash equilibrium with threshold probability $\rho(n)$, each player must believe that each other player chooses not to volunteer with probability $q(n) = G(\rho(n))$.

A player for whom $c/b = \rho(n)$ will be indifferent between volunteering and not volunteering. This implies that $H(q(n), n) = \rho(n)$ where $G(q(n), n) = \rho(n)$. It follows that the equilibrium value of $q(n)$ can be found by solving the equation

$$H(q(n), n) = n\left(\frac{(1 - q(n))}{q(n)}\right)\left(\frac{q(n)^n}{1 - q(n)^n}\right) = G^{-1}(q(n)) \tag{16}$$

for $q(n)$. As we show in Lemma 2 of the Appendix, $H(q, n)$ is an decreasing function of

We can do this numerically for some examples.

For the uniform distribution on $[0, b]$ we have $F(\rho) = \frac{1}{b}\rho$ and $G(\rho) = 1 - \frac{1}{b}\rho$. Thus $G^{-1}(q) = \frac{b}{1-q}$.

For the Pareto distribution with support $[\ell, \infty)$, $G(\rho) = \ell^{\alpha}\rho^{-\alpha}$ and $G^{-1}(q) = \ell q^{-\frac{1}{\alpha}}$.

For the exponential distribution, $G(\rho) = e^{-\alpha\rho}$ and so $G^{-1}(q) = \frac{-\ln q}{\alpha}$.

Maybe we can learn something interesting by solving these functions numerically for a few alternative parameter choices.

Appendix

A1: Proof of Proposition 4

Two lemmas are used in the proof of Proposition 4.

Lemma 1. *Where all players in a coordinated Volunteer's dilemma have cost benefit ratio $c/b = \rho$, it must be that in a symmetric Nash equilibrium,*

$$n\left(\frac{q^n}{1-q^n}\right)\left(\frac{1-q}{q}\right) = \rho \tag{17}$$

Proof of Lemma 1. In a symmetric Nash equilibrium where each player chooses *Volunteer* with probability $p = 1 - q$ where $0 < q < 1$, it must be that the expected payoff for this mixed strategy is the same as that for the pure strategy *Not Volunteer*.

If each of the n players volunteers with probability $1 - q$, then there will be at least one volunteer with probability $1 - q^n$. Thus the expected benefit for each player is $b(1 - q^n)$. If there is at least one volunteer, that volunteer is equally likely to be any of the n players. Thus the expected cost to each player is $\frac{1}{n}c(1 - q^n)$. Therefore if each player volunteers with probability q, the expected payoff to each of them is

$$b(1-q^n) - \frac{c(1-q^n)}{n}.$$

If all other players volunteer with probability $1 - q$, then the expected payoff to *not Volunteer* is $b\left(1 - q^{n-1}\right)$.

Therefore in a symmetric Nash equilibrium it must be that

$$b(1-q^n) - \frac{c(1-q^n)}{n} = b\left(1-q^{n-1}\right) \tag{18}$$

Rearranging terms of Equation 18 and recalling that $\rho = c/b$, we see that this equation is equivalent to Equation 17. This proves Lemma 1 □

Lemma 2. *The function $H(q, n)$ has the following properties where $H(q, n)$ is defined by the equation*

$$H(q,n) \equiv n\left(\frac{q^n}{1-q^n}\right)\left(\frac{1-q}{q}\right)$$

with domain $q \in [0, 1)$ and n a positive integer:

(i) $H(q, n)$ is continuous and strictly increasing in q.

(ii) $H(0, n) = 0$ *and* $\lim_{q \to 1} H(q, n) = 1$

(iii) $H(q, n)$ *is strictly decreasing in* n.

Proof of Lemma 2. The function $H(q, n)$ is a product of continuous functions of q and hence continuous. Since for $0 \leq q < 1$,

$$\frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots q^{n-1}, \tag{19}$$

it must be that

$$\begin{aligned} H(q,n) &= \frac{nq^{n-1}}{1+q+q^2+\ldots q^{n-1}} \\ &= \frac{n}{q^{1-n}+q^{2-n}+\cdots+1} \end{aligned} \tag{20}$$

The denominator of this expression is seen to be a decreasing function of q and thus $H(q, n)$ is an increasing function of q for all positive integers n. This proves Claim (i) of the lemma. Claim (ii) is easily verified by examination of Equation 20.

To establish Claim (iii), note that Equation 20 implies that

$$\frac{n+1}{H(q,n+1)} = \frac{n}{H(q,n)} + q^{-n} \tag{21}$$

Equation 21 implies that

$$\begin{aligned} n\left(\frac{1}{H(q,n+1)} - \frac{1}{H(q,n)}\right) &= q^{-n} - \frac{1}{H(q,n+1)} \\ &= q^{-n} - \frac{q^{-n}+q^{1-n}+q^{2-n}+\cdots+1}{n} \\ &> 0 \end{aligned} \tag{22}$$

where the final inequality in Expression 22 follows from the fact that when $0 < q < 1$, it must be that $q^{-n} > q^{i-n}$ for all $i > 0$. The inequality in Expression 22 implies that $H(q, n+1) < H(q, n)$, which is the assertion in Claim (iii). □

With these lemmas in hand, we can prove Proposition 4.

Proof of Proposition 4. Since, by assumption, $0 < c < b$, we must have $0 < c/b = \rho < 1$. Lemma 2 informs us that $H(q, n)$ is a continuous increasing function of q on the interval $[0, 1)$ and that $H(0) = 0$ and $\lim_{q \to 1} = 1$. It follows that for any $\rho \in [0,1)$ and any integer $n \geq 2$, the equation $H(q,n) = \rho$ has a unique solution $q(\rho,n)$. Lemma 1 informs us that there is a unique symmetric mixed strategy Nash equilibrium in which each player refuses to volunteer with probability $q(\rho,n)$.

Since $H(q,n)$ is an increasing function of q and for all $\rho \in (0,1)$, $H\left(q(\rho,n)\right) = \rho$, it must be that $q(\rho,n)$ is an increasing function of ρ.

Suppose that $n' > n$. By definition of the function $q(\rho,n')$, it must be that $H(q(\rho,n'),n') = \rho = H(q(\rho,n),n)$. According to Lemma 2, $H(q,n)$ is a decreasing function of n. Therefore $H\left(q(\rho,n),n'\right) < H\left(q(\rho,n)\right) = \rho$. Since $H(q,n)$ is an increasing function of q, it must be that $q(\rho,n') > q(\rho,n)$. Therefore $q(\rho,n)$ is an increasing function of n.

□

A.2 Proof of Proposition 5.

Our proof uses the following lemmas.

Lemma 3. *If $0 < x < 1$ and n is a positive integer, then*

$$\frac{x(1-x^n)}{1-x^{n+1}} < \frac{n}{n+1} \tag{23}$$

Proof. Since

$$1 - x^n = (1-x)(1 + x + x^2 + \cdots + x^{n-1}) \tag{24}$$

and

$$1 - x^{n+1} = (1-x)(1 + x + x^2 + \cdots + x^n), \tag{25}$$

it follows that inequality 23 holds if and only if

$$\frac{x + x^2 + \cdots + x^n}{1 + x + x^2 + \cdots + x^n} < \frac{n}{n+1} \tag{26}$$

Inequality 26 is equivalent to

$$x + x^2 + \ldots x^n < n. \tag{27}$$

Since $0 < x < 1$, the inequality in 27 applies. It follows that Inequality 23 holds.

□

Lemma 4. *For all $q \in (0,1)$ and all integers $n \geq 1$, $H\left(q^{n/n+1}, n+1\right) < H(q,n)$.*

Proof of Lemma 4. Recalling that

$$H(q,n) = n\left(\frac{1-q}{q}\right)\left(\frac{q^n}{1-q^n}\right),$$

it must be that

$$H(q^{\frac{n}{n+1}}, n+1) = (n+1)\left(\frac{1-q^{\frac{n}{n+1}}}{q^{\frac{n}{n+1}}}\right)\left(\frac{q^n}{1-q^n}\right) \tag{28}$$

Therefore $H\left(q^{n/n+1}, n+1\right) < H(q,n)$ if and only if

$$(n+1)\left(\frac{1-q^{\frac{n}{n+1}}}{q^{\frac{n}{n+1}}}\right) < n\left(\frac{1-q}{q}\right) \tag{29}$$

or equvalently,

$$\frac{q^{\frac{1}{n+1}}\left(1-q^{\frac{n}{n+1}}\right)}{1-q} < \frac{n}{n+1} \tag{30}$$

Define $x = q^{\frac{1}{n+1}}$. Since $0 < q < 1$, it follows that $0 < x < 1$. Then Expression 30 can be written as

$$\frac{x(1-x^n)}{1-x^{n+1}} < \frac{n}{n+1} \tag{31}$$

Lemma 3 implies that this inequality holds and hence Inequality 30 is true. □

Proof of Proposition 5. The probability that nobody volunteers will be larger for a group of size $n+1$ than for a group of size n if $q(\rho, n+1)^{n+1} > q(\rho, n)^n$. This will be the case if $q(\rho, n+1) > q(\rho, n)^{n/(n+1)}$. The equilibrium conditions require that

$$H\left(q(\rho, n+1), n+1\right) = H\left(q(\rho, n), n\right) = \frac{b}{c}. \tag{32}$$

According to Lemma 4,

$$H\left(q(\rho, n)^{n/n+1}, n+1\right) < H(q(\rho, n), n). \tag{33}$$

Since, according to Lemma 2, $H(q, n+1)$ is an increasing function of q, it follows from Equations 32 and 33 that $q(\rho, n+1) > q(\rho, n)^{n/(n+1)}$ and hence $q(\rho, n+1)^{n+1} > q(\rho, n)^n$. This means that the probability that nobody volunteers increases with the number of players. □

A.3: Proof of Proposition 6

Lemma 5. *Where $\bar{Q}(\rho)$ is the limiting probability of no volunteers with large n, it must be that for all $\rho \in [0, 1)$*

$$\rho = \frac{-\bar{Q}(\rho)}{1 - \bar{Q}(\rho)} \ln \bar{Q}(\rho). \tag{34}$$

Proof of Lemma 5. Since $Q(n, \rho) = q(n, \rho)^n$, the equilibrium condition in Equation 17 can be written equivalently as

$$\rho = \frac{n\left(1 - Q(n, \rho)^{\frac{1}{n}}\right) Q(n, \rho)^{\frac{n-1}{n}}}{1 - Q(n, \rho)} \tag{35}$$

which implies that

$$(1 - Q(n, \rho))\, \rho = n\left(Q(n, \rho)^{1-\frac{1}{n}} - Q(n, \rho)\right) \tag{36}$$

Making a change of variables $t = 1/n$, Equation 36 can be written as

$$\left(1 - Q\left(\frac{1}{t}, \rho\right)\right)\rho = \frac{1}{t}\left(Q\left(\frac{1}{t}, \rho\right)^{1-t} - Q\left(\frac{1}{t}, \rho\right)\right). \tag{37}$$

Taking limits as $t \to 0$ and recalling that $\bar{Q}(\rho) = \lim_{n\to\infty} Q(n, \rho)$, yields the equation:

$$\left(1 - \bar{Q}(\rho)\right)\rho = \lim_{t\to 0} \frac{\bar{Q}(\rho)^{1-t} - \bar{Q}(\rho)}{t} \tag{38}$$

Where we define the function $g(x) = Q^x$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\bar{Q}(\rho)^{1-t} - \bar{Q}(\rho)}{t} &= lim_{t\to 0} \frac{g(1-t) - g(1)}{t} \\
&= -lim_{t\to 0} \frac{g(1-t) - g(1)}{-t} \\
&= -g'(1) \\
&= -\bar{Q}(\rho) \ln \bar{Q}(\rho)
\end{aligned} \tag{39}$$

From Equations 38 and 39 it follows that

$$\rho = \frac{-\bar{Q}(\rho)}{1 - \bar{Q}(\bar{\rho})} \ln \bar{Q}(\rho) \tag{40}$$

□

Lemma 6. *The function* $\rho(Q)$ *defined on domain* $(0,1)$ *by*

$$\rho(Q) = \frac{-Q}{1-Q}\ln Q \tag{41}$$

is strictly increasing and maps onto the interval $(0,1)$*.*

Proof of Lemma 6. To show that $\rho(Q)$ is strictly increasing on $(0,1)$, note that

$$\rho'(Q) = -\frac{1}{(1-Q)^2}\left(1 - Q + \ln Q\right). \tag{42}$$

Therefore $\rho'(Q) < 0$ if $1 - Q + \ln Q < 0$ for all $Q \in (0,1)$. Let $g(Q) = 1 - Q + \ln Q$. Then $g(1) = 0$ and $g'(Q) = -1 + \frac{1}{Q} > 0$ for all $x \in (0,1)$. It follows that $g(Q) < 0$ for all $Q \in (0,1)$. Therefore $\rho'(Q) > 0$ for all $Q \in (0,1)$ and so $\rho(Q)$ is strictly increasing on $(0,1)$.

To show that $\rho(Q)$ maps onto the entire interval $(0,1)$, we apply L'Hospital's rule to find that:

$$\lim_{Q\to 0} \rho(Q) = 0 \tag{43}$$

and

$$\lim_{Q\to 1} \rho(Q) = 1. \tag{44}$$

Thus we have:

$$\begin{aligned} \lim_{Q\to 0} \rho(Q) &= -\lim_{Q\to 0}\left(\frac{1}{1-Q}\right)\lim_{Q\to 0}\left(\frac{\ln Q}{\frac{1}{Q}}\right) \\ &= -1 \times \lim_{Q\to 0}\left(\frac{\frac{1}{Q}}{\frac{-1}{Q^2}}\right) \\ &= \lim_{Q\to 0} Q \\ &= 0. \end{aligned} \tag{45}$$

$$\begin{aligned} \lim_{Q\to 1} \rho(Q) &= -\lim_{Q\to 1}\left(\frac{Q\ln Q}{1-Q}\right) \\ &= -\lim_{Q\to 1}\frac{\ln Q + 1}{-1} \\ &= 1 \end{aligned} \tag{46}$$

Therefore the continuous function $\rho(Q)$ maps onto the entire interval $(0,1)$. Since $\rho(Q)$ is strictly increasing, this mapping must be one-to-one. □

Proof of Proposition 6 . Lemma 6 implies that $\bar{\rho}^{-1}(\rho)$ is a well-defined increasing function on the interval $(0, 1)$. Lemma 5 implies that $\bar{r}ho\,(Q(\rho)) = \rho$ and hence $Q(\rho) = \bar{\rho}^{-1}(\rho)$.

$\square$

References

[1] Theodore C. Bergstrom, Rodney Garratt, and Damien Sheehan-Connor. One chance in a million: Altruism and the bone marrow registry. *American Economic Review*, 99(4):1309–1334, September 2009.

[2] Andreas Diekmann. Volunteer's dilemma. *The Journal of Conflict Resolution*, 29(4):605–610, December 1985.

[3] Jeroen Weesie. Asymmetry and timing in the volunteer's dilemma. *The Journal of Conflict Resolution*, 37(3):569–590, September 1993.